ANALYZING n SAMPLES
OF 2 OBSERVATIONS EACH

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I. Introduction.

When data are scarce, it is common to combine small samples from a number of sources considered to be reasonably similar. When sample sizes are extremely small, testing the assumption of similarity of sources is often only attempted by subjective means. This paper provides a method to add quantitative risk assessments to the study of this assumption, using two observations per sample.

In addition to general compatibility testing of the sources using modified versions of Westenberg's Interquartile Range Test, and the Westenberg-Mood Median Test, a new hypothesis test has been developed to aid in identifying whether one (or more) of the sources of data provides a substantially larger or smaller set of values due to its underlying population.

Because the probabilistic risk assessments provided here address a situation so commonly found in analyzing military operations as well as in test and evaluation, details are provided to simplify the implementation of this methodology. A major goal here has been that power analyses be described in terms meaningful to the user and the decision maker. The new hypothesis test makes use of simulation-aided power analyses.

The tests for general comparability, using modified Westenberg tests, were first introduced in reference 2. The FORTRAN code for these tests is given here in Appendix I. This program involves straight forward binomial probabilities. The first format statement explains the variables and the null and alternative hypotheses. It is written in terms of the interquartile range of the combined sample, but could easily be written in terms of being above or below the median of the combined sample. The null hypothesis (H_o) and the alternative (H_1) are repeated here. "H_o: Each sample has at least 100 x RA percent chance of having one observation inside and one observation outside the interquartile range. H_1: Each sample has at least 100 x RB percent chance of having both observations fall together either inside or outside." The null hypotheses for the Westenberg tests thus indicate a general compatibility among the data sources. If a set of data made up of pairs of observations from a number of sources appear to be reasonably homogeneous as judged by these tests, however, it may still be that one, or perhaps a few pairs of observations may have been drawn from a source very different from the underlying population for the majority of the pairs of observations. Therefore, a new test is needed due to the inadequacy of the modified Westenberg tests to discern such a situation.

In order to determine whether a pair of observations may have values appreciably larger, or smaller than the other observations, the probability of having both members of that pair of observations be among the largest, or smallest in the combined sample should be investigated. This is accomplished here in the Length of Initial Run (LIR) Test. In this test, the larger value in each pair of observations is labeled "A," while the smaller of the observations from that source is labeled "B." If no pair of observations is drawn

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from an underlying population considerably different from the others (in particular if no pair of observations is drawn from an underlying population whose location is considerably larger, or smaller than the others) then the length of the run of A's in the combined sample before the first B and the length of the run of B's before the first A should not be too short. (Note that run lengths can vary from one to n.)

II. Mathematical Theory of LIR Test.

Under the null hypothesis, that all pairs of observations were drawn from the same or identical populations, the length of the initial run of A's is identically distributed as the length of the initial run of B's. This discussion will only be couched in terms of the initial run of A's.

The probability of having an "A" as the largest value in the combined sample is unity. The probability of having an "A" in the second largest position, under H_o, is the number of observations not included in the sample pair that the first A came from $(2n-2)$, divided by the total remaining number of observations available $(2n-1)$. The probability of having a third A in a row is the previous probability multiplied by the number of observations not included in either sample pair that the first two A's came from $(2n-4)$, divided by the total remaining number of observations available $(2n-2)$. Therefore, the probability (under H_o) of having at least r of the A's before the first B is:

$$\left(\frac{2n-2}{2n-1} \right) \left(\frac{2n-4}{2n-2} \right) \ \left(\frac{2n-6}{2n-3} \right) \cdot \cdot \cdot \cdot \left(\frac{2n-2r+2}{2n-r+1} \right)$$

If exactly r of the A's preceed the first B, the probability of this occurrence (under H_o) is the above expression multiplied by the probability that the r+1th largest value is a B. This would be the number of B values whose corresponding A value is among the members of the initial run (r), divided by the total remaining number of observations $(2n-r)$. Simplifying, therefore, the probability (under H_o) of having an initial run of length r is:

$$\frac{(2n-2)!!(2n-r-1)!r}{(2n-2r)!!(2n-1)!}$$, where $k!! = k(k-2)(k-4) \cdot \cdot \cdot \cdot$ (stopping at 2 if k is even, or 1 if k is odd).

This further simplifies to

$$\frac{(n-1)!2^{n-1}(2n-r-1)!r}{(n-r)!2^{n-r}(2n-1)!} = \frac{(n-1)!2^{r-1}(2n-r-1)!r}{(n-r)!(2n-1)!}$$

In general, if N is the number of observations per sample and n is the number of samples, then the probability of an initial run of length r is:

$$\frac{(n-1)!N^{r-1}(Nn-r-1)!(N-1)r}{(n-r)!(Nn-1)!}$$

Simulations could be used for alternative hypotheses and for irregular numbers of observations. This paper, however, is concerned with N=2 observations per sample.

Simulations (see Appendix III) were used to determine the relative frequency distribution of initial run lengths under various alternative hypotheses. Each alternative studied assumed one pair of observations to be taken from one underlying population and all others taken from a second underlying population, with a few exceptions for sensitivity study purposes. The accuracy of the simulations was examined in several manners. First, both underlying populations were set identical and the results compared to the frequency distribution for the null hypothesis. Agreement here demonstrated that the closed form solution for the null distribution is correct and also that the simulation was accurate to approximately three significant digits using 20,000 replications for the cases of interest shown in Appendix II, also under H_o.

However, under any alternative hypothesis, accuracy of the simulation is degraded due to the fact that the distributional forms which the pairs of observations are being drawn from are not exactly what they have been represented to be. Table I, however, provides a set of chi-square "poorness" of fit tests which show that, in the case investigated there, the distribution is almost exactly as was represented. (Similar results were obtained using other distributions.) Table II is used to demonstrate the small differences in resulting output when inputs are varied to degrees that were found unlikely to actually occur. (Note that the differences found in Table II were of only approximately the same magnitude as in Table I.) From this, it is generally concluded that only two significant digits should be used from the relative and cumulative relative frequency outputs.

In addition to the type of validation shown above, the simulation results were compared to a closed form solution for the probability of a run of length one when one pair of observations is drawn from one distribution and all others from a second distribution. In order for there to be a run of length one, both observations from the same pair must be the two largest (or smallest) observations in the combined sample. Therefore, if p is the probability of a run of length one, and we are investigating the initial run of A's, and only one pair of observations is drawn from one distribution with all others drawn from a second distribution,

$$p = \int_{x=-\infty}^{\infty} g_{2,m,m}(x) \int_{t=x}^{\infty} g_{1,2,1}(t)dtdx$$

$$+ \frac{1}{m-1} \int_{x=-\infty}^{\infty} g_{1,2,2}(x) \int_{t=x}^{\infty} g_{2,m,m-1}(t)dtdx \ .$$

		True N(12,1)		VS		Random Nos. Generated From A N(12,1)		
CELL #	#OBS	CELL#	#OBS	CELL#	#OBS	CELL#	#OBS
1	370	26	410	51	417	76	405
2	375	27	428	52	402	77	403
3	402	28	382	53	398	78	405
4	369	29	413	54	381	79	360
5	412	30	386	55	447	80	361
6	376	31	400	56	385	81	393
7	407	32	396	57	393	82	388
8	424	33	426	58	375	83	408
9	413	34	386	59	404	84	381
10	381	35	433	60	401	85	388
11	377	36	380	61	419	86	404
12	380	37	407	62	371	87	388
13	394	38	418	63	414	88	408
14	381	39	393	64	400	89	421
15	382	40	407	65	415	90	411
16	364	41	370	66	396	91	422
17	388	42	428	67	415	92	397
18	398	43	411	68	429	93	395
19	423	44	395	69	389	94	398
20	427	45	405	70	424	95	394
21	381	46	347	71	417	96	414
22	398	47	413	72	406	97	429
23	393	48	445	73	439	98	392
24	394	49	415	74	404	99	420
25	370	50	392	75	392	100	417

THE CHI-SQUARE VALUE FOR A N(12,1)
USING 40000 GENERATED RANDOM NUMBERS
TESTED AGAINST A N(12,1)
IS 94.56

TABLE Ib

	True N(12,1)		VS		Random Nos. Generated From A N(12,0.95)		
CELL #	#OBS	CELL#	#OBS	CELL#	#OBS	CELL#	#OBS
1	282	26	385	51	449	76	437
2	295	27	426	52	415	77	390
3	317	28	430	53	418	78	349
4	344	29	403	54	402	79	387
5	343	30	424	55	468	80	410
6	372	31	409	56	395	81	394
7	361	32	424	57	426	82	399
8	380	33	424	58	385	83	393
9	390	34	409	59	435	84	392
10	402	35	433	60	417	85	389
11	378	36	417	61	444	86	401
12	362	37	412	62	387	87	409
13	370	38	445	63	413	88	406
14	396	39	424	64	454	89	394
15	382	40	403	65	399	90	414
16	383	41	426	66	439	91	375
17	362	42	401	67	435	92	368
18	369	43	458	68	416	93	382
19	412	44	399	69	432	94	366
20	436	45	432	70	441	95	383
21	438	46	387	71	424	96	366
22	385	47	418	72	451	97	367
23	388	48	460	73	418	98	335
24	416	49	442	74	398	99	354
25	396	50	415	75	414	100	300

THE CHI-SQUARE VALUE FOR A N(12,0.95)
USING 40000 GENERATED RANDOM NUMBERS
TESTED AGAINST A N(12,1)
IS   302.45

TABLE Ic

	True N(12,1)	VS		Random Nos. Generated From A N(12,1.05)			
CELL #	#OBS	CELL#	#OBS	CELL#	#OBS	CELL#	#OBS
1	495	26	418	51	402	76	393
2	443	27	366	52	371	77	384
3	448	28	407	53	388	78	398
4	445	29	378	54	372	79	398
5	412	30	386	55	398	80	359
6	442	31	367	56	396	81	342
7	444	32	427	57	366	82	385
8	428	33	356	58	372	83	401
9	399	34	423	59	370	84	392
10	395	35	352	60	383	85	380
11	396	36	386	61	387	86	398
12	399	37	423	62	400	87	396
13	378	38	385	63	358	88	402
14	384	39	362	64	368	89	422
15	354	40	385	65	422	90	417
16	404	41	365	66	385	91	433
17	411	42	403	67	376	92	433
18	411	43	394	68	421	93	418
19	413	44	387	69	381	94	416
20	383	45	360	70	382	95	427
21	373	46	345	71	404	96	424
22	392	47	392	72	411	97	473
23	392	48	427	73	392	98	479
24	368	49	391	74	431	99	472
25	391	50	380	75	391	100	566

THE CHI-SQUARE VALUE FOR A N(12,1.05)
USING 40000 GENERATED RANDOM NUMBERS
TESTED AGAINST A N(12,1)
IS 277.28

TABLE IIa

Input distributions are N(12.00, 1.00) and N(10.00, 1.00). Number of samples from each distribution is 1 and 9 respectively. The random number seed for this run is 65557.

Test for the Length of the Initial Run of A's before the first B.

Number of replications: 20000

Length of Run:	Observed Frequency:	Relative Frequency:	Cumulative Frequency:
1	7275	0.363750	0.363750
2	4391	0.219550	0.583300
3	2980	0.149000	0.732300
4	2123	0.106150	0.838450
5	1534	0.076700	0.915150
6	857	0.042850	0.958000
7	517	0.025850	0.983850
8	246	0.012300	0.996130
9	67	0.003350	0.999500
10	10	0.000500	1.000000

TABLE IIb

Input distributions are N(12.00, 0.95) and N(10.00, 1.00). Number of samples from each distribution is 1 and 9 respectively. The random number seed for this run is 65557.

Test for the Length of the Initial Run of A's before the first B.

Number of replications: 20000

Length of Run:	Observed Frequency:	Relative Frequency:	Cumulative Frequency:
1	7406	0.370300	0.370300
2	4480	0.224000	0.594300
3	3018	0.150900	0.745200
4	2134	0.106700	0.851900
5	1419	0.070950	0.922850
6	806	0.040300	0.963150
7	469	0.023450	0.986600
8	203	0.010150	0.996750
9	55	0.002750	0.999500
10	10	0.000500	1.000000

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TABLE IIc

Input distributions are N(12.00, 1.05) and N(10.00, 1.00). Number of samples from each distribution is 1 and 9 respectively. The random number seed for this run is 65557.

Test for the Length of the Initial Run of A's before the first B.

Number of replications: 20000

Length of Run:	Observed Frequency:	Relative Frequency:	Cumulative Frequency:
1	7141	0.357050	0.357050
2	4273	0.213650	0.570700
3	2969	0.148450	0.719150
4	2119	0.105950	0.825100
5	1611	0.080550	0.905650
6	958	0.047900	0.953550
7	556	0.027800	0.981350
8	284	0.014200	0.995550
9	78	0.003900	0.999450
10	11	0.000550	1.000000

where $g_{1,2,k}(x)$ is the distribution of the $k\,\underline{\text{th}}$ order statistic out of 2 observations in distribution 1 (distribution 1 is, in general, the distribution of larger location if the initial run of A's is being investigated); and $g_{2,m,k}$ is the distribution of the $k\,\underline{\text{th}}$ order statistic out of m for distribution 2. Since there are n-1 pairs of observations taken from distribution 2, m = 2(n-1) = 2n-2, and therefore:

$$\frac{1}{m-1} = \frac{1}{2n-2-1} = \frac{1}{2n-3}$$

The expression for p is based on the fact that if both observations of a given pair have larger values than any other observation in the combined sample, then the B value associated with the largest A has to be larger than all other 2n-2 values.

Once again, use of this validation technique supported the conclusion that two significant digits should be used in the results.

If $f_1(x)$ is the density function for distribution 1, and $f_2(x)$ for distribution 2, an approximation can be made for p when the number of observations drawn from each distribution are equal, or nearly equal, and very small. This approximation will be very poor unless the assumptions are enforced. In general, however, the calculations are much easier than those in the previous expression. In this case the approximation is as follows:

$$\phi = \int_{x=-\infty}^{\infty} f_2(x) \int_{t=x}^{\infty} f_1(t)dtdx; \quad \Phi = \phi/(1-\phi)$$

$$p \approx \left(\frac{\nu\Phi}{\nu\Phi+2n-\nu}\right)\left(\frac{\Phi}{(\nu-1)\Phi+2n-\nu}\right) + \left(\frac{2n-\nu}{\nu\Phi+2n-\nu}\right)\left(\frac{1}{\nu\Phi+2n-\nu-1}\right)$$

where ν is the number of observations taken from the first distribution. ϕ is the probability that if one observation were drawn from each of the two distributions, the observation from distribution 1 would have a larger value. The approximation is therefore a weighted counting procedure which does not fully account for the shapes of the true distributions of interest.

The tables of Appendix II provide power information for a variety of cases when the number of samples (of size two each) is 5, 10, 20, and 50. The alternative hypotheses could have more than two underlying distributions (up to n) but two are sufficient to illustrate what is being investigated here. The importance of this test is to determine the likelihood of having one (or possibly more than one) pair of observations drawn from an underlying population which is substantially different from the underlying population from which the rest of the observations were drawn.

If sample sizes are large, it is possible to reject H_o when the truth is close enough to H_o for practical purposes, unless a specific H_1 is used for a power analysis. When sample sizes are small, as in the case here, one may fail to reject H_o when the truth is not close to H_o. Power analyses would help by completing the quantification of the problem. Without a power analysis, a hypothesis test is only half completed. Null and alternative hypotheses work in pairs analogously to confidence limits. In the present situation, a power analysis is very important due to the unconventional nature of the problem. (It is interesting to note that when the number of samples [of size two each] increases, the power level at a given significance level remains apparently approximately constant [see Appendiix II].)

This paper makes use of a much neglected application for simulation. Simulation can be used as a check for a closed form solution when the development of such a solution was subtle, in addition to handling situations where a closed form solution is difficult if not impossible.

III. Example.

Suppose that ten processes (or items of equipment, etc.) are to be examined for a certain trait and that it is expected that they will all behave similarly for that trait. Also, suppose that the expense involved in studying those processes is great, or that for some other practical reason, the number of observations per process (item, scenario, etc.) must be kept extremely small. If two observations each are used, a combined sample size of 20 is obtained. Whether or not these samples should be combined would then be open to examination. In addition to any subjective arguments, the modified Westenberg tests and the LIR test should be applied to assist in this examination. Suppose (for use in the LIR test) that the larger value in each pair of observations is labelled "A," and the smaller values labelled "B." Subscripts "1"-"10" could be used to denote which process is being represented. (This will be used in the modified Westenberg tests.) Suppose that when the values for each of these observations are ranked from largest to smallest the following result is obtained:

$$A_8 \; B_8 \; A_7 \; A_2 \; A_1 \;\bigg|\; A_4 \; A_5 \; A_6 \; A_{10} \; A_3 \;\bigg|\; B_2 \; A_9 \; B_7 \; B_4 \; B_1 \;\bigg|\; B_6 \; B_{10} \; B_3 \; B_9 \; B_5$$

The number of times that a pair of observations are both found on the same side of the median, NZ, is 2. Also, the number of times that a pair of observations are both found either inside or outside the interquartile range is NZ=2. Table III shows results taken from the program of Appendix I. PA is the probability level of the test associated with H_o, and PB is the probability level associated with H_1. An examination of this table shows that there is no good reason, based on this ranked data and aside from subjective arguments, to conclude that these samples should not be combined, but actually there is good reason to conclude that such a combination is advisable. However, the LIR test can be used to show that even though general compatibility appears evident, the

TABLE III

```
RUN WC
   HO:  EACH SAMPLE HAS AT LEAST 100xRA% CHANCE OF HAVING
        ONE OBSERVATION INSIDE AND ONE OBSERVATION OUTSIDE
         THE INTERQUARTILE RANGE.

   H1:  EACH SAMPLE HAS AT LEAST 100xRB% CHANCE OF HAVING
        BOTH OBSERVATIONS FALL TOGETHER
        EITHER INSIDE OR OUTSIDE




    INPUTS ARE:
    NS, THE NUMBER OF SAMPLE
    NL, THE NUMBER OF ZEROES
    RA AND RB

    THE NUMBER OF ZEROES IS THE NUMBER OF SAMPLES WHOSE TWO OBSERVATIONS
    ARE FOUND TOGETHER


     ENTER NS,NZ,RA,RB
10,2,0.95,0.50
PA=0.2262      PB=0.1875
Do you wish to run the test again
Enter "Y" or "N".
Y


     ENTER NS,NZ,RA,RB
10,2,0.50,0.50
PA=0.9688   .   PB=0.1875
Do you wish to run the test again?
Enter "Y" or "N".
Y


     ENTER NS,NZ,RA,RB
10,2,0.75,0.25
PA=0.7627      PB=0.6328
Do you wish to run the test again?
Enter "Y" or "N".
N
FORTRAN STOP
```

H₀: 10 pairs of observations drawn from the same distribution

H₁: one pair drawn from a N(50,4) and nine pair drawn from a N(40,4) if looking at initial run of A's, or vice versa if looking at initial run of B's.

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GRAPH B

H_0: 10 pairs of observations drawn from the same distribution

H_1: one pair from $N(48,4)$ and nine pair from $N(40,4)$ (for A's, or vice versa for B's)

GRAPH C

H_0: 10 pairs of observations drawn from the same distribution

H_1: 5 pairs drawn from $N(50,4)$ and 5 pairs from $N(40,4)$

initial run of A's is extremely short (one) and this indicates that sample 8 may provide values too large in comparison to the other values. The LIR test shows that if the null hypothesis (that all pairs of observations were drawn from the same underlying distribution) is true, the probability of an initial run of length one is 0.053 when the number of samples (n) is 10. This is the probability of a Type I error. However, if the null hypothesis is "accepted" here, the probability of a Type II error (for any alternative) is unity since no amount of evidence against H_o would then suffice. If the values of the

observations here suggest that graph A is the result of a reasonable alternative hypothesis, then the power against that alternative is of interest and is approximately 0.56. Against the alternative used to arrive at graph B, the power is 0.36, and the power from graph C would be 0.12. Note, however, that the alternative shown here against which the power is lowest is for a situation better investigated by the tests of Appendix I, program WC, originally found in reference 2. Note also that even if all 20 observations here came from the same distribution, that is a small sample to use to determine the form of that distribution. Therefore, other alternatives involving distributional forms other than normality may be needed to complete the analysis.

Suppose B_8 and A_7 were not as shown, but exchanged in position. The values of NZ would remain the same, but the length of the initial run of A's would now be two. If H_o is rejected when the observed run length is 2, the probability of a Type I error is 0.158. Against the first, second, and third alternatives mentioned above, the probability of a Type II error would be 0.44, 0.64, and 0.88, respectively, and the power would be 0.76, 0.58, and 0.34, also respectively.

If a run of length 3 is considered, these figures are, respective to the order given above, 0.307, 0.24, 0.42, 0.66, 0.87, 0.73, and 0.62.

Considering the above, when n=10 it could be deemed reasonable to reject H_o when the run length is 2 or less, and "accept" it when it is 3 or more.

In conclusion, if some of the observations in the example given earlier are shifted in rank it may affect one or more or perhaps none of these tests. Also, if the modified Westenberg tests greatly discourage the combining of the samples, then the LIR test will probably not be very useful. In using the modified Westenberg tests, if it is not possible to divide the observations into groups of equal size (above and below the median and inside and outside of the interquartile range), then apply the tests shifting the observation(s) which are in question from one possible grouping to the other and average the results obtained. Finally, because the Bernoulli trials in the modified Westenberg tests used here are not truly independent, these tests are approximate. Also, some RA and RB values may be inappropriate. For example, in Table III, RA could never be 0.95; however, it is used to illustrate the strong conclusiveness of these particular results. A study of the null hypothesis indicates that for RA = 0.5, the binomial distribution used for this

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test has thicker "tails" than warranted. (The exact null distribution is described by

$$\Sigma\{\frac{(NS!/2^{NZ/2})^2}{(2NS)!/2^{NS}} \cdot \frac{NS!/(NS-NZ)!}{[(NZ/2)!]^2}\}.)$$

Also, the true null distribution is skewed to the right, but not appreciably when NS exceeds 50. For smaller numbers of samples, WB 1 from reference 2 is of use. The advantages in using program WC are that understandable alternative hypotheses can be shown for decision making, and a large number of sample pairs can be handled with very little computer time. All that is really needed is a table of the cumulative binomial distribution. This test should be considered "quick and dirty" as a preliminary to the implementation of the LIR test. The LIR test is an exact test and is easily and meaningfully applied.

APPENDIX I

FORTRAN CODE FOR Modified Westenberg Test
(Designed for n pairs of observations)

This program was referred to as "WC" in reference 2. Note that unlike the other modified Westenberg tests of reference 2, which are exact, this one is an approximate test.

```
        CHARACTER*4 ANSWER
        WRITE(6,1100)
1100    FORMAT(5X,'HO:  EACH SAMPLE HAS AT LEAST 100xRA% CHANCE',
      / ' OF HAVING',/,9X,'ONE OBSERVATION INSIDE AND ONE'
      / ' OBSERVATION OUTSIDE',/,9X,' THE INTERQUARTILE RANGE.',
      / //,5X,'H1:  EACH SAMPLE HAS AT LEAST 100xRB% CHANCE',
      / ' OF HAVING',/,9X,'BOTH OBSERVATIONS FALL TOGETHER',
      / /,9X,'EITHER INSIDE OR OUTSIDE',///////,5X,'INPUTS ARE: ',
      / /,5X,'NS, THE NUMBER OF SAMPLES',/,
      / 5X,'NZ, THE NUMBER OF ZEROES',
      / /,5X,'RA AND RB',//,5X,'THE NUMBER OF ZEROES IS THE NUMBER OF',
      / ' SAMPLES WHOSE TWO OBSERVATIONS',/,5X,'ARE FOUND TOGETHER')
1000    WRITE(5,999)
 999    FORMAT(//,5X,'ENTER NS,NZ,RA,RB')
        READ(5,*)NS,NZ,RA,RB
        I=NS/2
        J=NZ/2
        PA=0
        PB=0
        K=0
   2    X=I
        Y=K
        W=0.5*I
        IF(Y.GE.W)GOTO 100
   3    P=X
        P=P/(X-Y)
        T=0
   5    T=T+1
        P=P*(X-T)
        P=P/(X-Y-T)
        L=X-Y-T
        IF(L.GT.1)GOTO 5
        IF(K.GT.J)GOTO 6
        PB=PB+P*((RB**K)*((1-RB)**(I-K)))
        K=K+1
        IF(K.LE.J)GOTO 2
        K=K-1
   6    PA=PA+P*(((1-RA)**K)*(RA**(I-K)))
        K=K+1
        IF(K-I)2,2,2000
 100    Y=X-Y
        GOTO 3
2000    WRITE(6,2001)PA,PB
2001    FORMAT(1X,'PA=',F6.4,5X,'PB=',F6.4)
        WRITE(6,2002)
2002    FORMAT(T2,'Do you wish to run the test again?',/,
      / T2,'Enter "Y" or "N".')
        READ(5,2003)ANSWER
2003    FORMAT(A)
        IF(ANSWER.EQ.'Y')GOTO 1000
2010    STOP
        END
```

APPENDIX II

POWER TABLES FOR LIR TEST

The distributions from which the input is to be drawn for each of the alternative hypotheses are shown here followed by histograms of the relative frequency distributions for these alternatives. In all cases here, both of the distributions from which the samples are drawn are of the same type but with parameters which differ in some respect. The samples could have been drawn from totally different distributional forms, and more than two such distributions could have been used (up to n); however, what is used here is sufficient to demonstrate this test under conditions which illustrate its usefulness.

In the case of the normal distribution, when standard deviations are the same, the symmetric nature makes the relative frequency distribution for the initial run of A's the same as that for the initial run of B's, if the number of pairs of observations from each of the two input distributions is interchanged. This is true in all cases where symmetric input distributions of equal variance are used. Also, when two symmetric input distributions with the same location, but unequal variances are used, this principle applies. Whenever this occurs, the output relative frequency distributions here are written in terms of the initial run of A's.

The Church-Harris-Downton (C-H-D) method of testing the probability of motor case rupture in missile testing, see reference 1, makes use of a statistic related to ϕ (shown in Section II here). This statistic is $(\mu_1 - \mu_2)/(\sigma_1^2 + \sigma_2^2)^{1/2}$, where the subscripts "1" and "2" refer to the two input distributions. For any of the tables involving two normal distributions, if μ_1, μ_2, σ_1, and σ_2 are changed such that the above statistic remains constant, then the output relative frequency distributions given here are applicable.

In the case of the gamma input distributions, whenever the β's (scale parameters) are both multiplied by the same factor, the output distributions are still applicable. For triangular input distributions, if all parameters are added to, subtracted from, divided or multiplied by the same number, the output distributions will not be changed.

The tables given here are for two normal distributions, two gamma distributions, two triangular distributions and finally, two beta distributions. The parameters were picked, in many cases, such that the power against the alternatives was approximately 0.5 when the significance level was approximately 0.1 to 0.15. This occurs for H_o, as shown in the following table:

# of samples of size 2 each	run length under H_o	(probability level under H_o)
5	1	~11%
10	2	~15%
20	3	~15%
50 {	4	~10%
	5	~15%

The following tables provide a variety of examples of alternative hypotheses and results obtained using them. It is hoped that this appendix is sufficient to provide a working knowledge of the power of this test to its users. When any specific alternative which the user is interested in investigating does not appear here, and the user does not wish to spend the time to get the programs of Appendix III to run at a convenient facility, it is hoped that the results can be interpolated from results provided here. Note that following each graph of the distributions from which the observations are hypothesized (H_1) to have been drawn, only relative frequency distributions are provided. The hollow bar graphs show the relative frequency distribution under H_o.
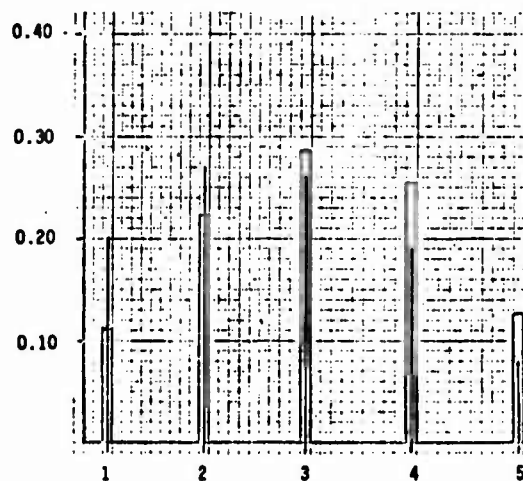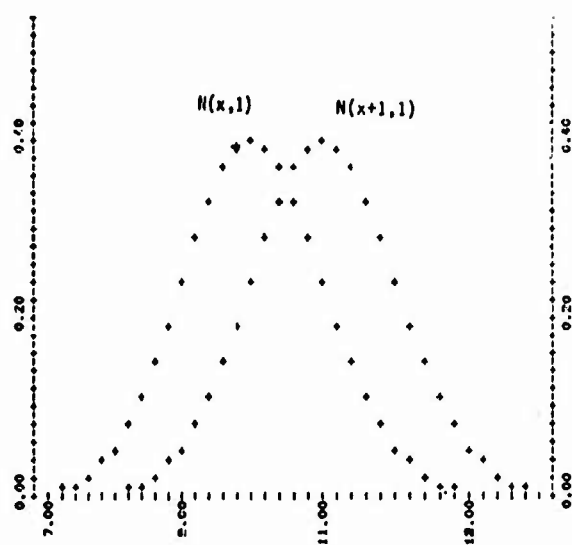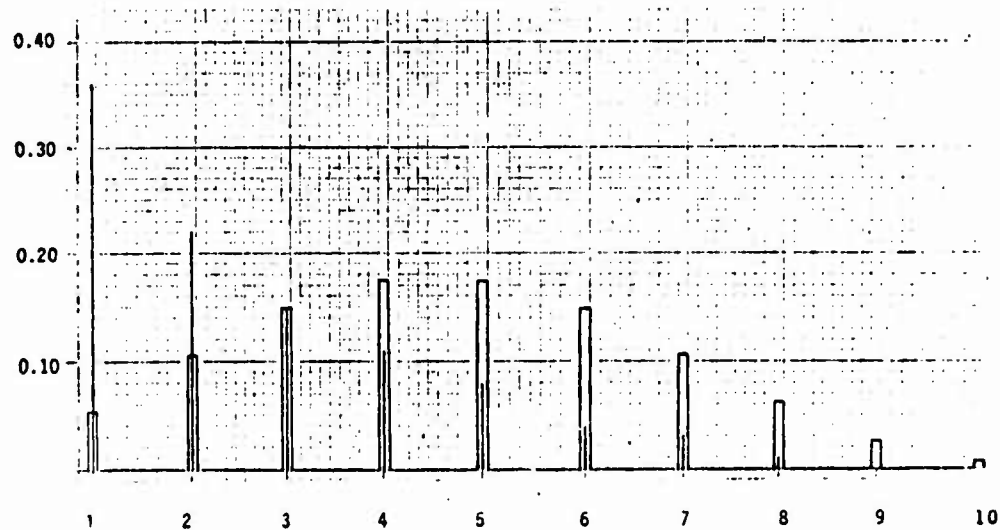
N(x,1)   N(x+1,1)

H_0: 5 pairs of observations drawn from the same distribution

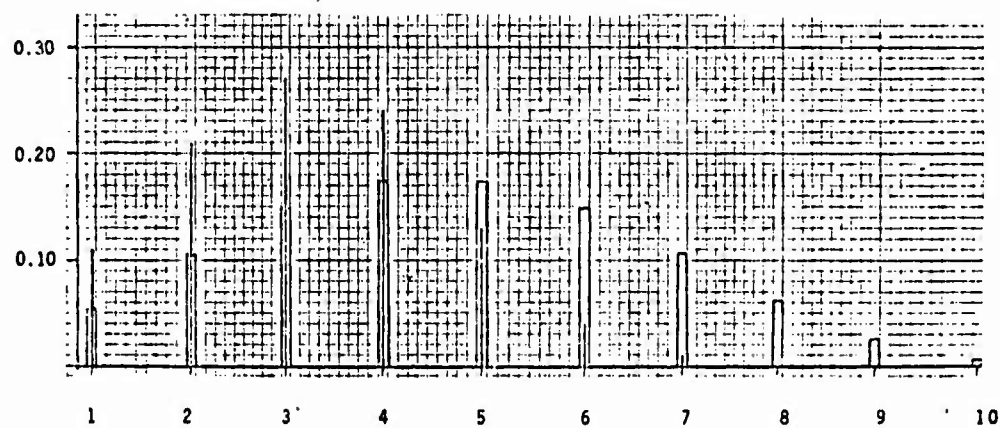H_1: 4 pairs from $N(x,1)$ and 1 pair from $N(x+1,1)$
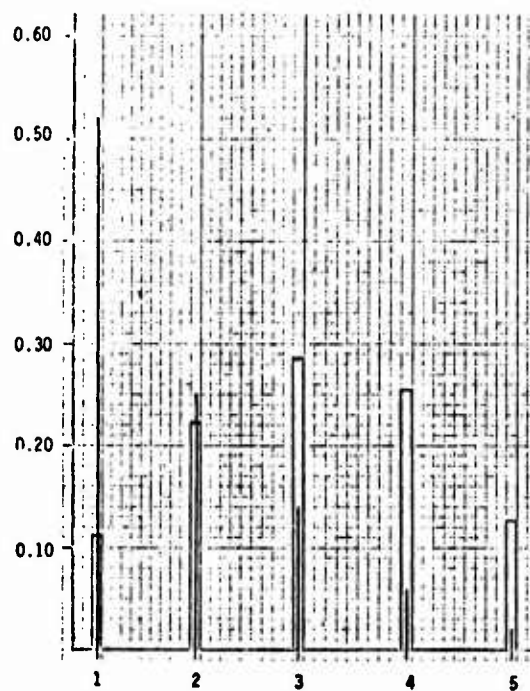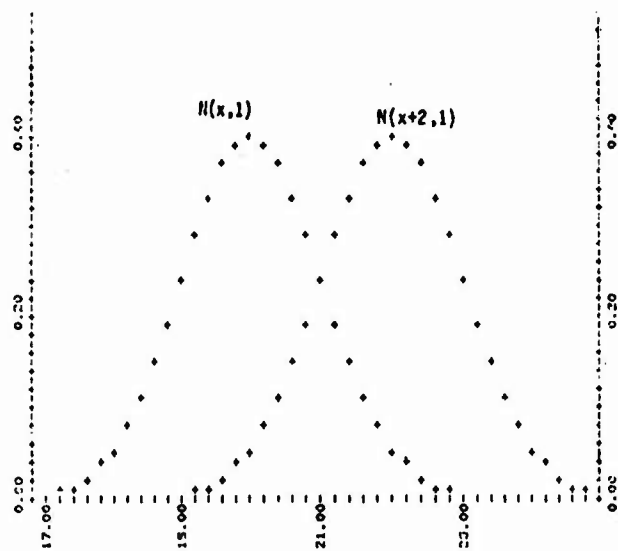
45

H_0: 5 pairs of observations drawn from the same distribution

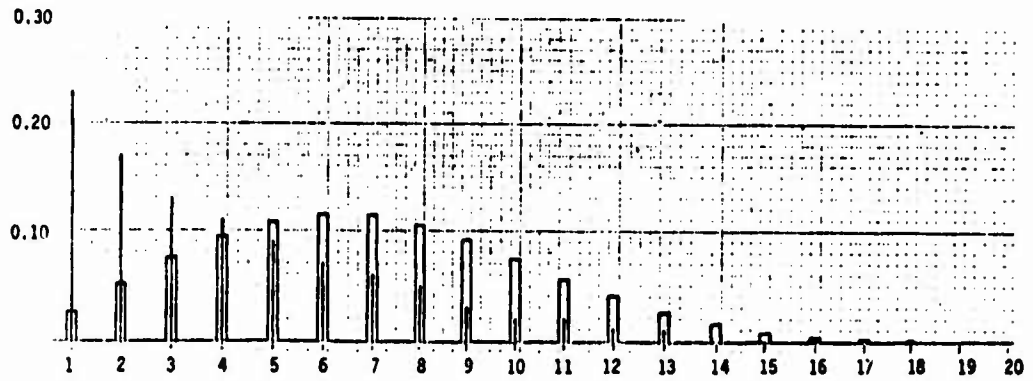H_1: 4 pairs from $N(x,1)$ and 1 pair from $N(x+2,1)$

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H_0: 10 pairs of observations drawn from the same distribution

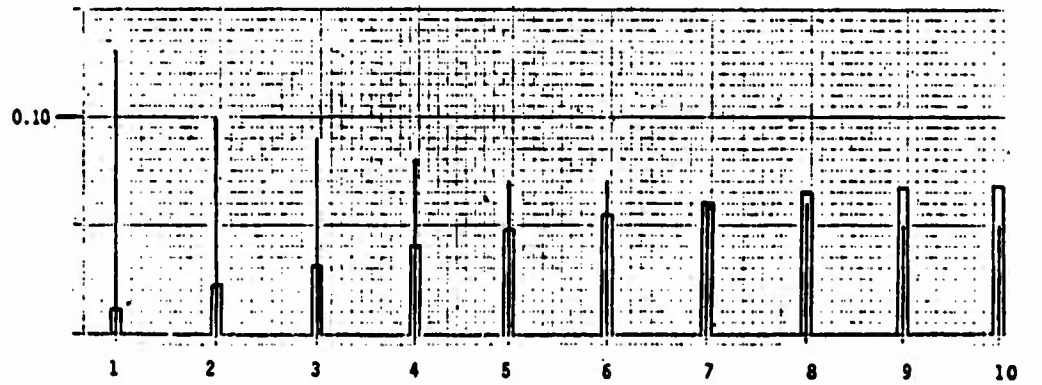H_1: 9 pairs from $N(x,1)$ and 1 pair from $N(x+2,1)$

H_0: 10 pairs of observations drawn from the same distribution

H_1: 5 pairs from $N(x,1)$ and 5 pairs from $N(x+2,1)$

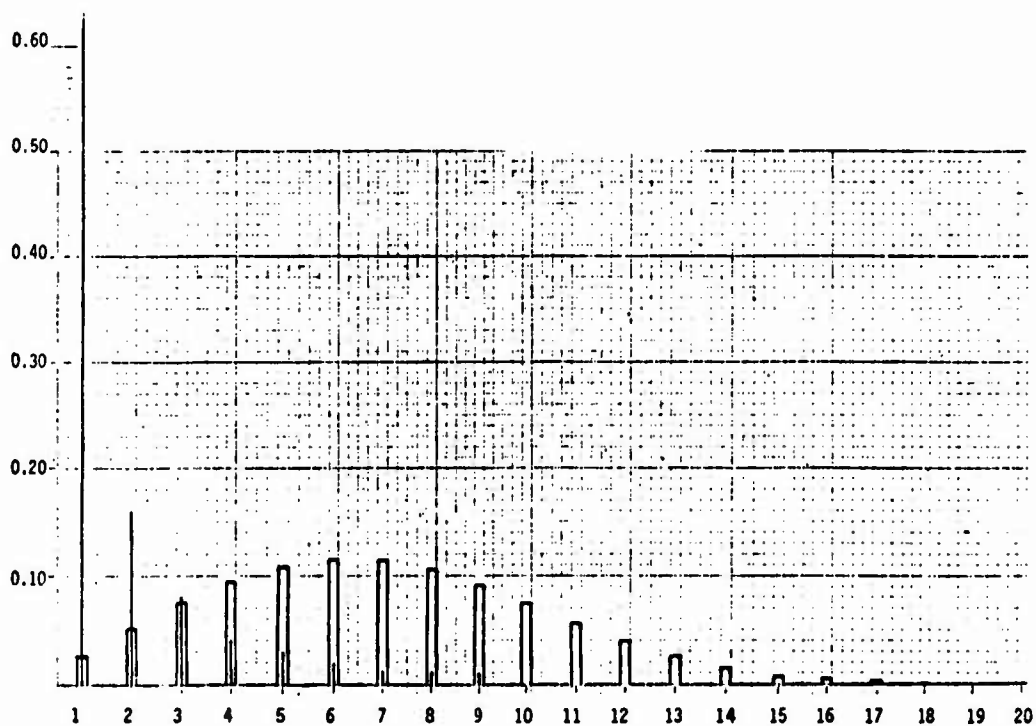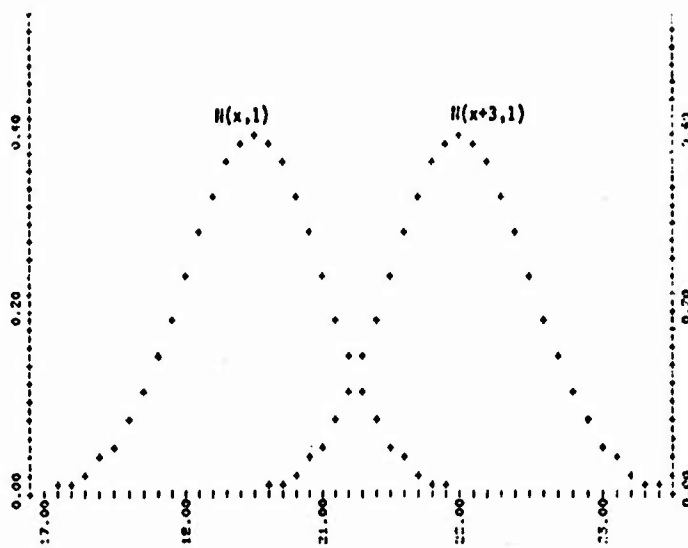H_0: 20 pairs of observations drawn from the same distribution

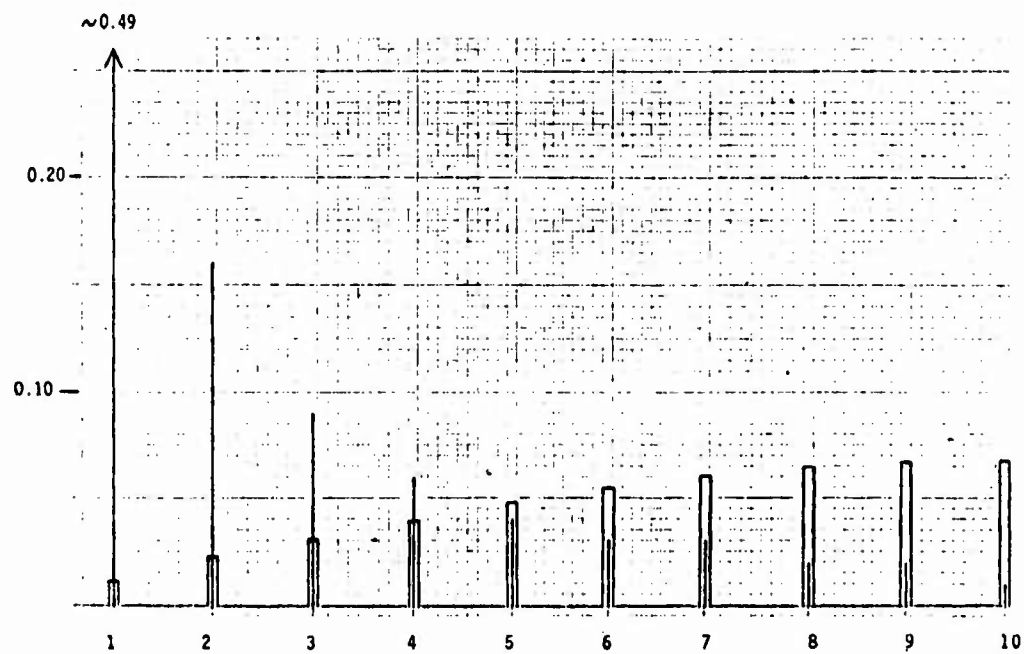H_1: 19 pairs from $N(x,1)$ and 1 pair from $N(x+2,1)$

H_0: 50 pairs of observations drawn from the same distribution

H_1: 49 pairs from $N(x,1)$ and 1 pair from $N(x+2,1)$

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N(x,1) N(x+3,1)

0.40

0.20

0.00

17.00   19.00   21.00   21.00   23.00

0.40

0.20

0.00

0.60

0.50

0.40

0.30

0.20

0.10

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

H_0: 20 pairs of observations drawn from the same distribution

H_1: 19 pairs from $N(x,1)$ and 1 pair from $N(x+3,1)$

H_0: 50 pairs of observations drawn from the same distribution

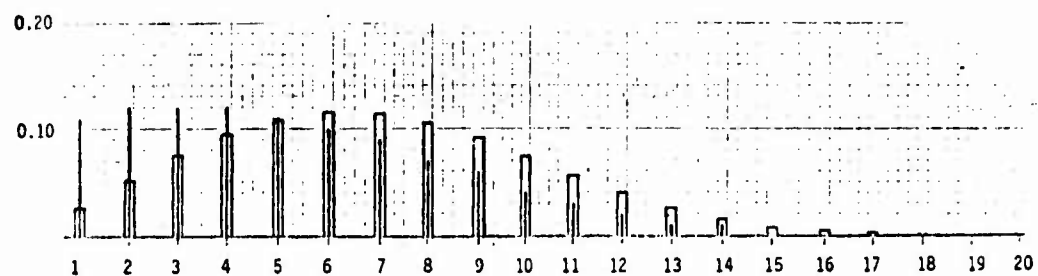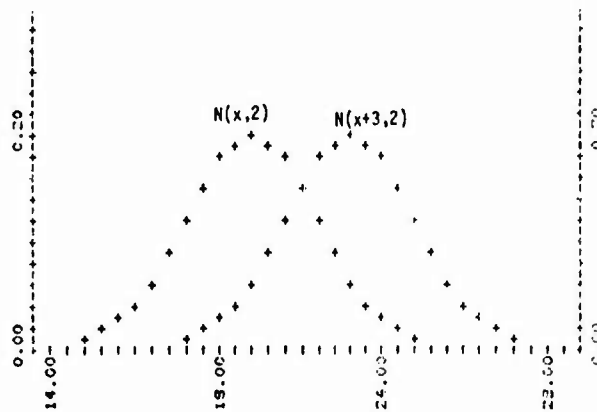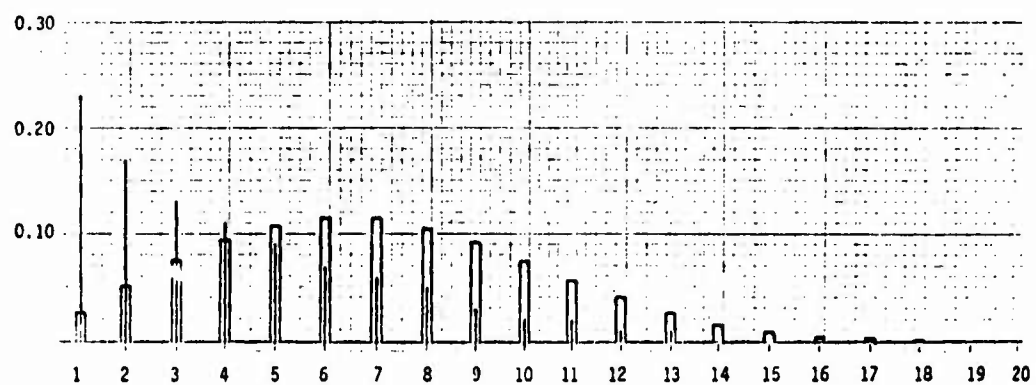H_1: 49 pairs from $N(x,1)$ and 1 pair from $N(x+3,1)$

H_0: 20 pairs of observations drawn from the same distribution

H_1: 19 pairs from $N(x,2)$ and 1 pair from $N(x+3,2)$

N(x,2)   N(x+4,2)

0.20

0.00

14.00   19.00   24.00   29.00

0.30

0.20

0.10

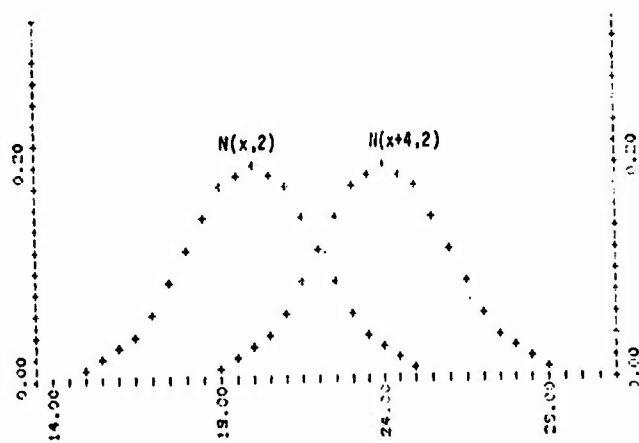1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

H_0: 20 pairs of observations drawn from the same distribution

H_1: 19 pairs from N(x,2) and 1 pair from N(x+4,2)

H_0: 20 pairs of observations drawn from the same distribution

H_1: 19 pairs from $N(x,2)$ and 1 pair from $N(x+5,2)$

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H_0: 20 pairs of observations drawn from the same distribution

H_1: 10 pairs from $N(x,2)$ and 10 pairs from $N(x+5,2)$

N(x,4)   N(x+8,4)

0.20   0.20   0.00   0.00

33.00   49.00   58.00   59.00

0.40

0.30

0.20

0.10

1   2   3   4   5   6   7   8   9   10

H_0: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from $N(x,4)$ and 1 pair from $N(x+3,4)$

N(x,4)   N(x+10,4)

H₀: 10 pairs of observations drawn from the same distribution

H₁: 9 pairs from N(x,4) and 1 pair from N(x+10,4)

H_o: 10 pairs of observations drawn from the same distribution

H_1: 5 pairs from $N(x,4)$ and 5 pairs from $N(x+10,4)$

H_0: 10 pairs of observations drawn from the same distribution
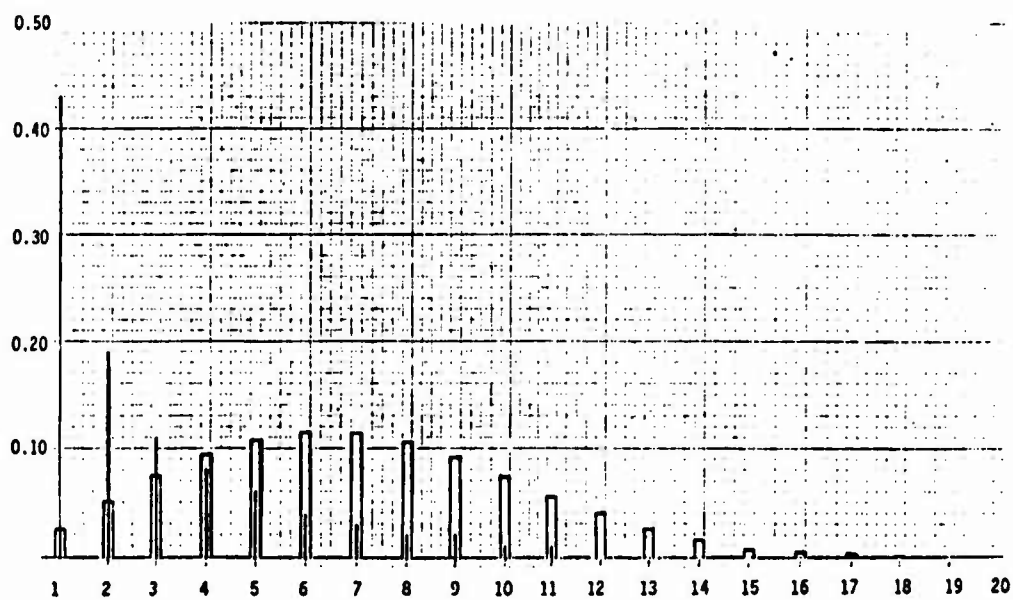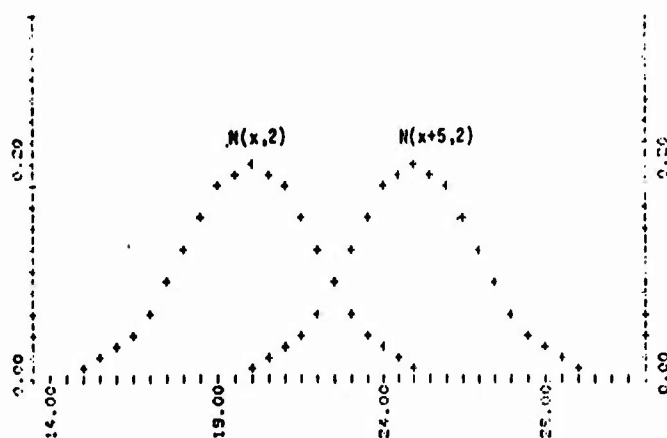
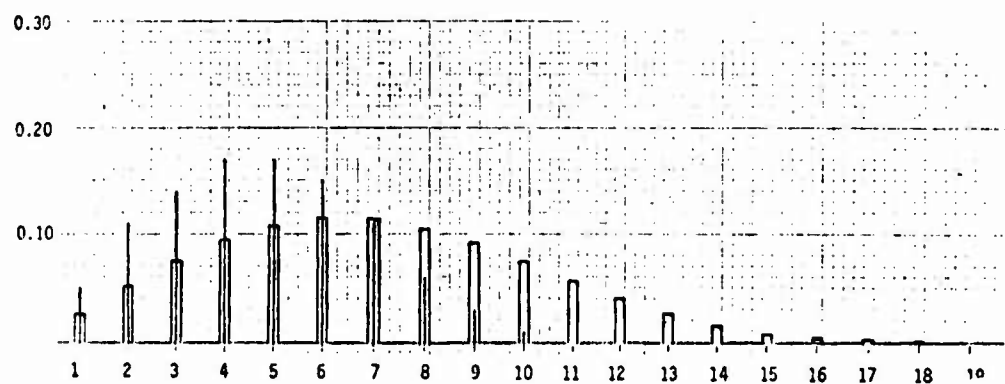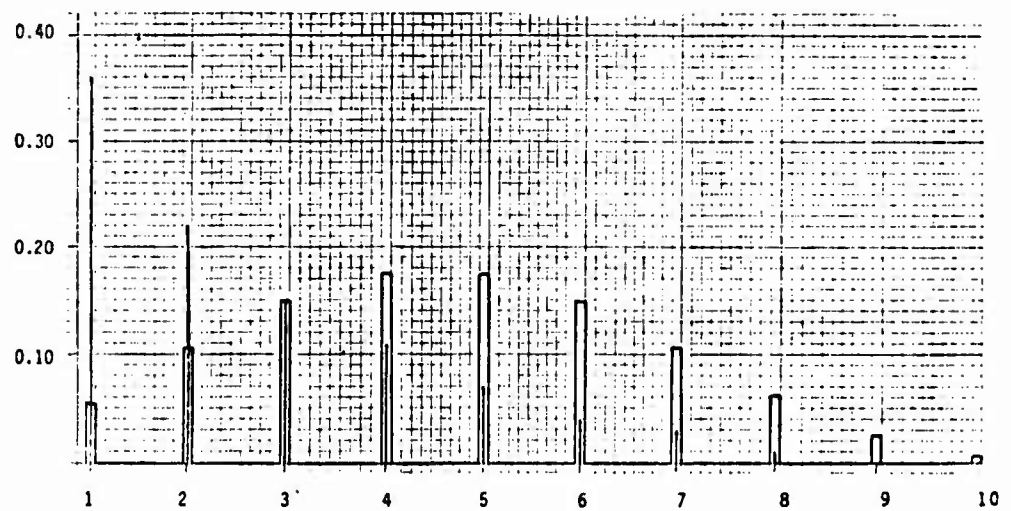H_1: 9 pairs from $N(x,y)$ and 1 pair from $N(x,4y)$

N(x,y)

N(x,10y)

H_0: 10 pairs of observations drawn from the same distribution

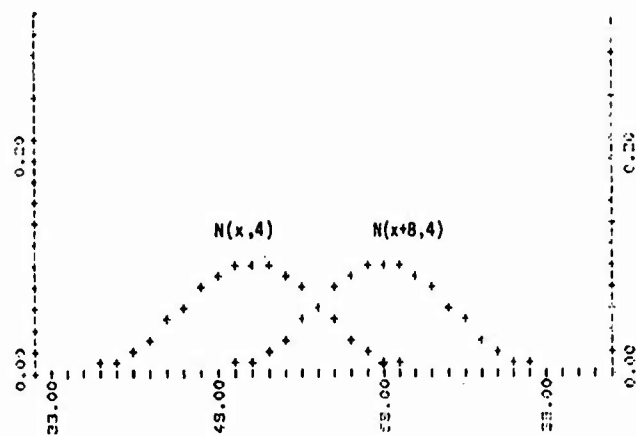H_1: 9 pairs from N(x,y) and 1 pair from N(x,10y)
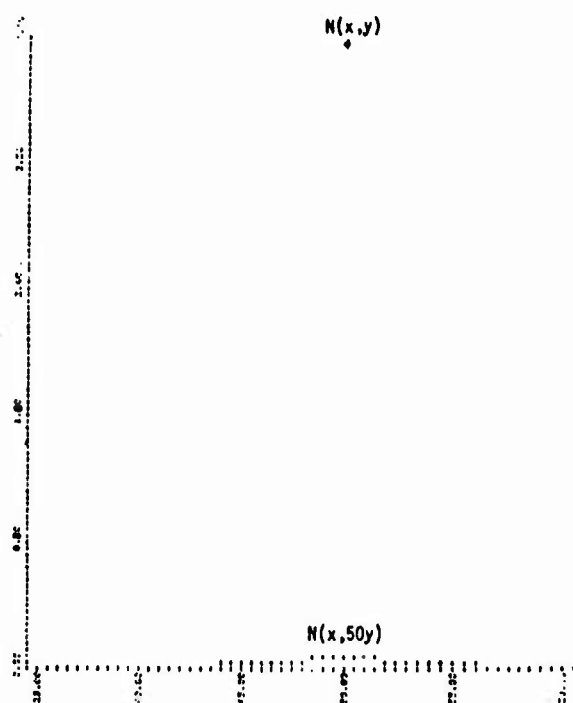
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N(x,y)

N(x,50y)

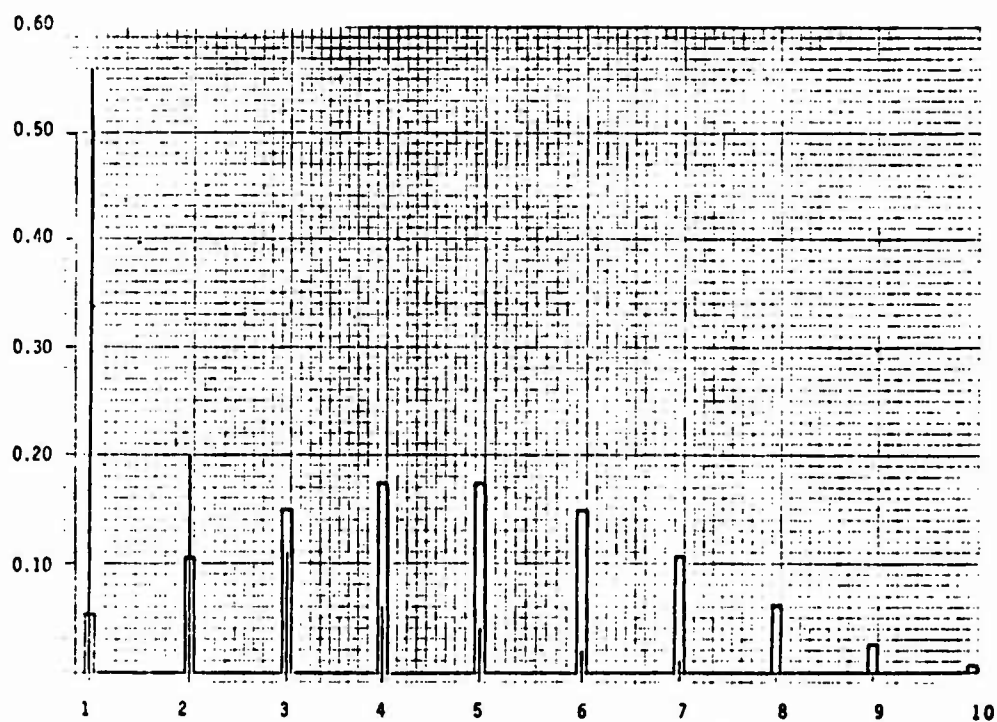H_0: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from N(x,y) and 1 pair from N(x,50Y)

60
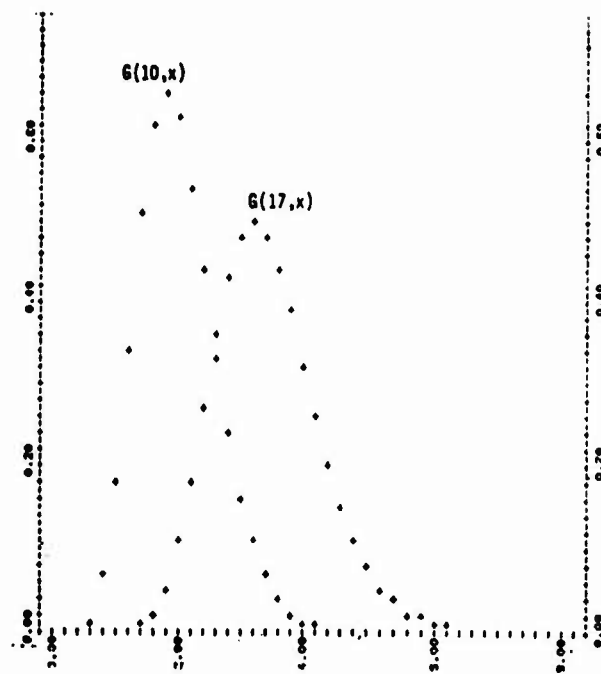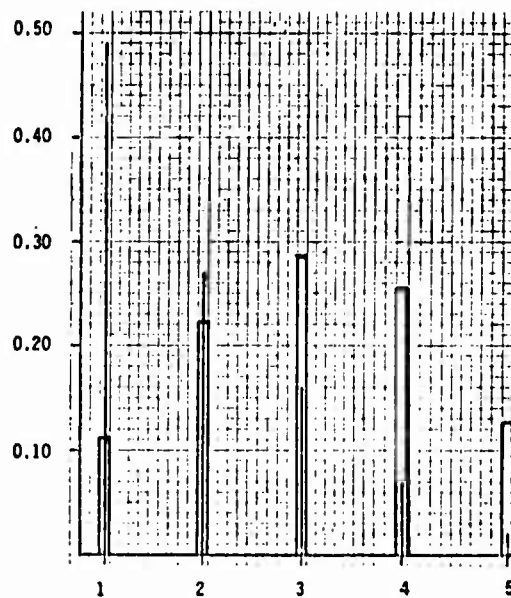
H_o: 5 pairs of observations drawn from the same distribution

H_1: 4 pairs from $G(10,x)$ and 1 pair from $G(17,x)$ (for LIR of A's)
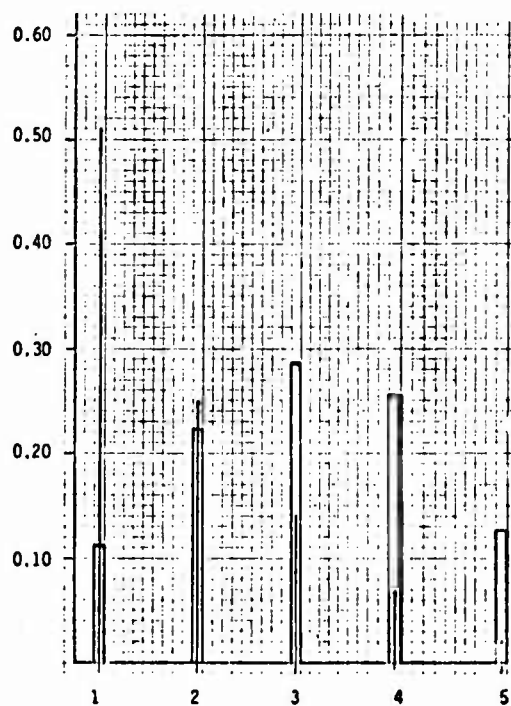
H_o: 5 pairs of observations drawn from the same distribution

: 4 pairs from $G(17,x)$ and 1 pair from $G(10,)$ (for LIR of B's)
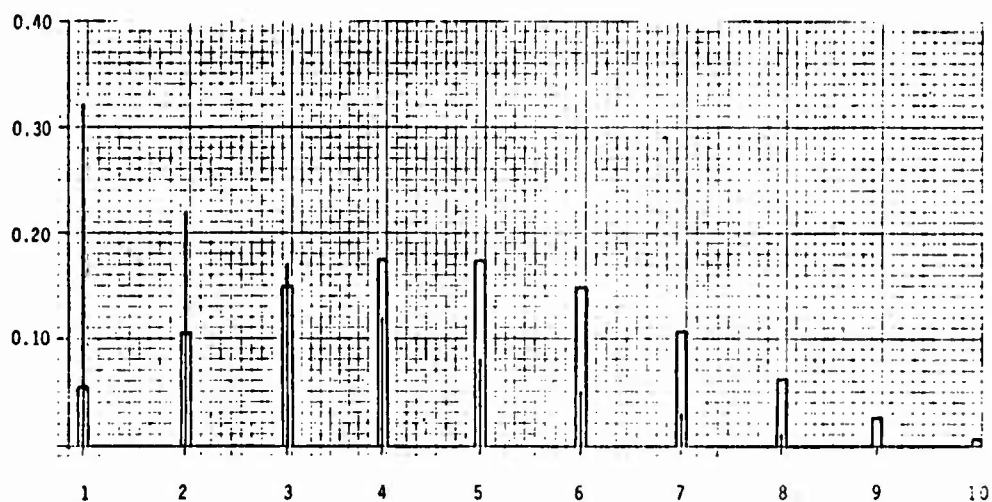
H_0: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from $G(10,x)$ and 1 pair from $G(17,x)$ (for LIR of A's)

H_0: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from $G(17,x)$ and 1 pair from $G(10,x)$ (for LIR of B's)

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H_o: 10 pairs of observations drawn from the same distribution

H_1: 5 pairs from $G(17,x)$ and 5 pair from $G(10,x)$ (for LIR of A's)

H_o: 10 pairs of observations drawn from the same distribution

H_1: 5 pairs from $G(17,x)$ and 5 pair from $G(10,x)$ (for LIR of B's)
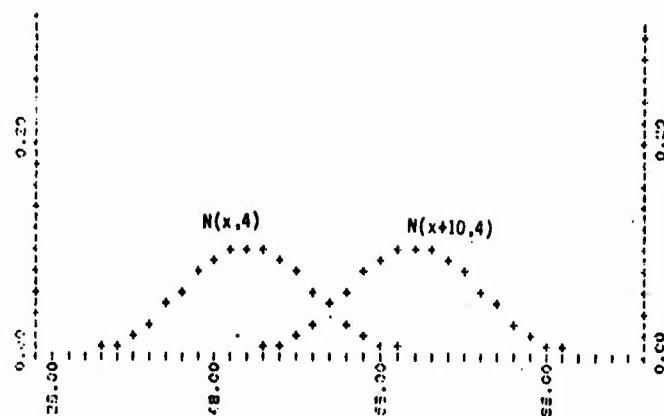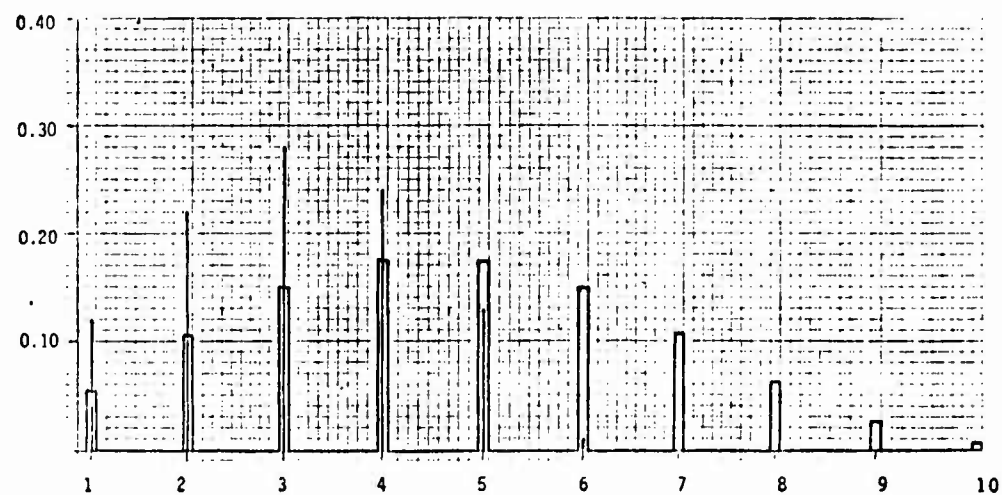
64
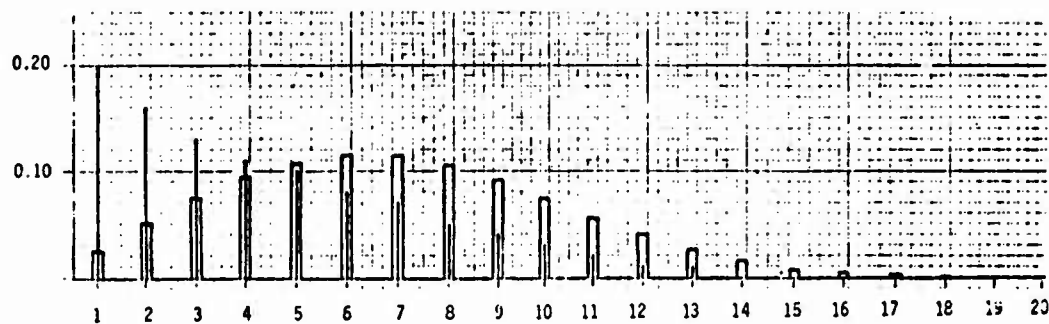
H_0: 20 pairs of observations drawn from the same distribution

H_1: 19 pairs from $G(10,x)$ and 1 pair from $G(17,x)$ - A's
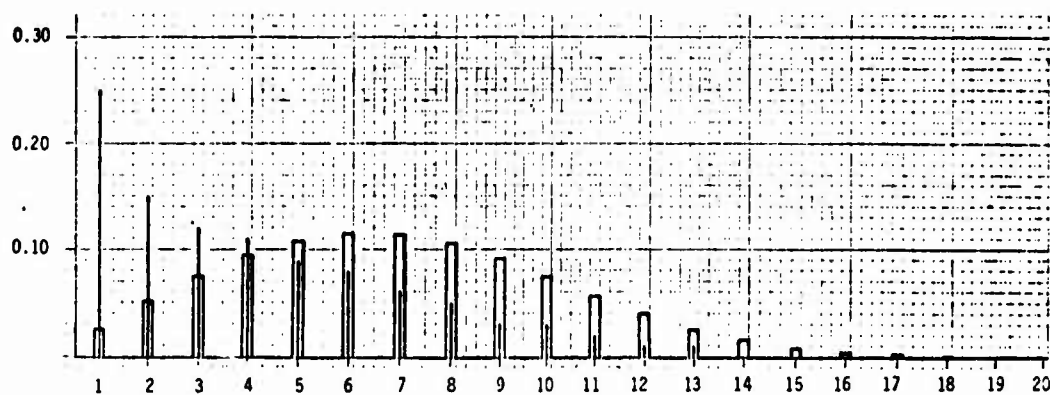
H_0: 20 pairs of observations drawn from the same distribution

H_1: 19 pairs from $G(17,x)$ and 1 pair from $G(10,x)$ - B's
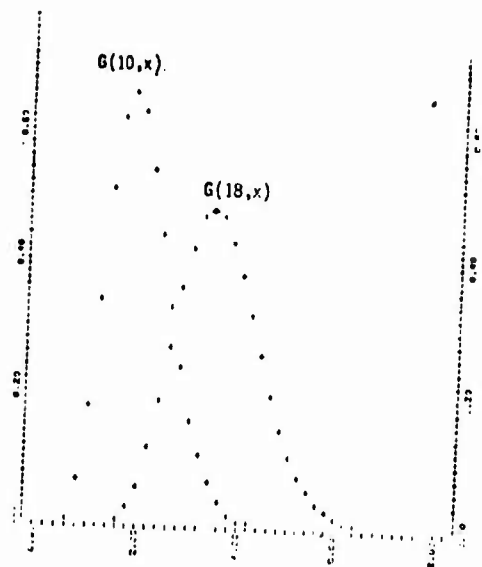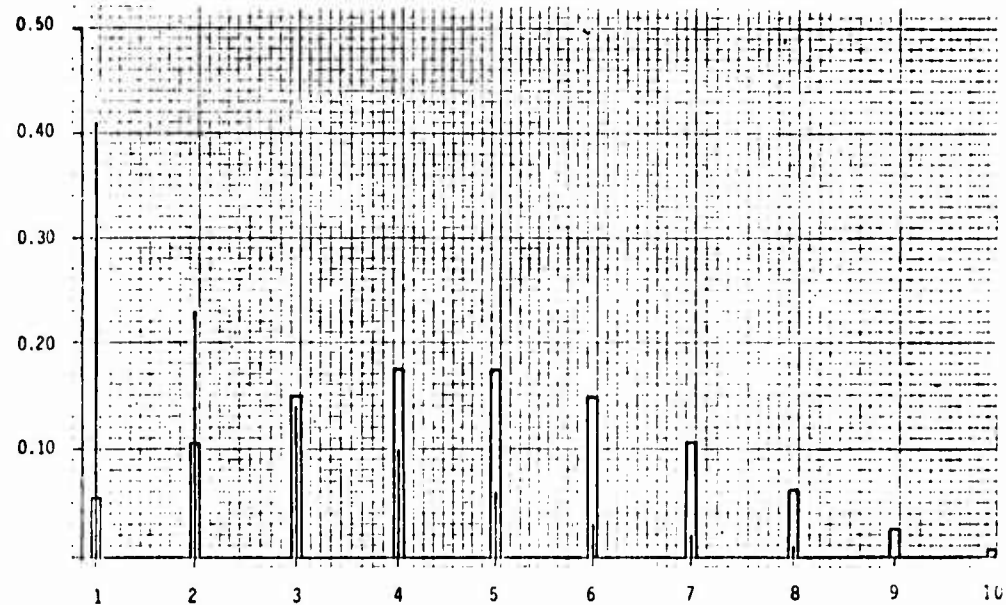
65

H_0: 10 pairs of observations drawn from the same distribution

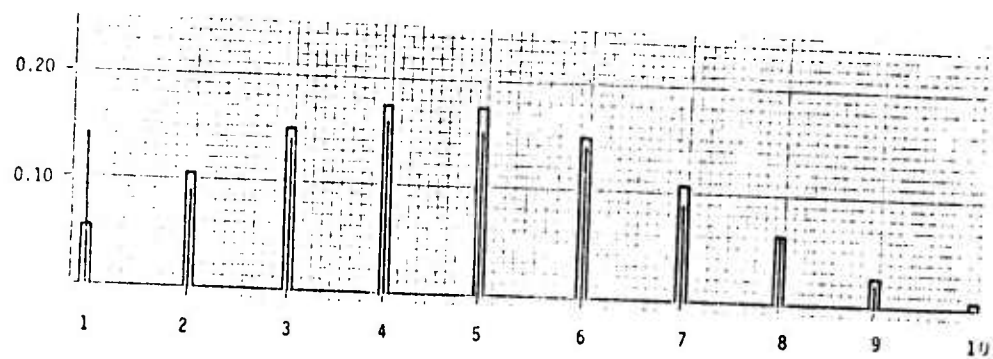H_1: 9 pairs from $G(10,x)$ and 1 pair from $G(18,x)$ - A's
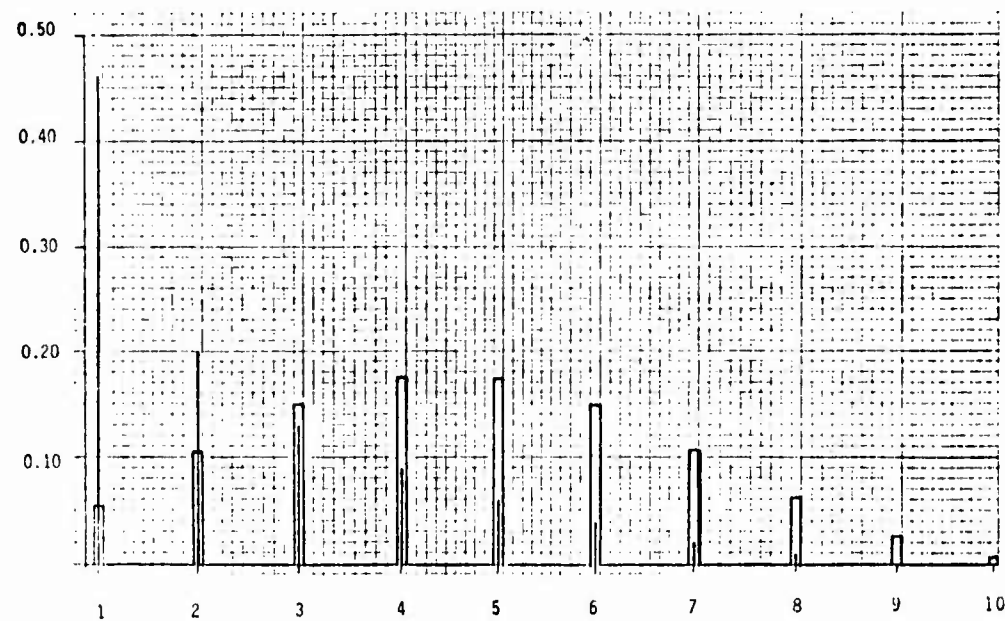
H_0: 10 pairs of observations drawn from the same distribution

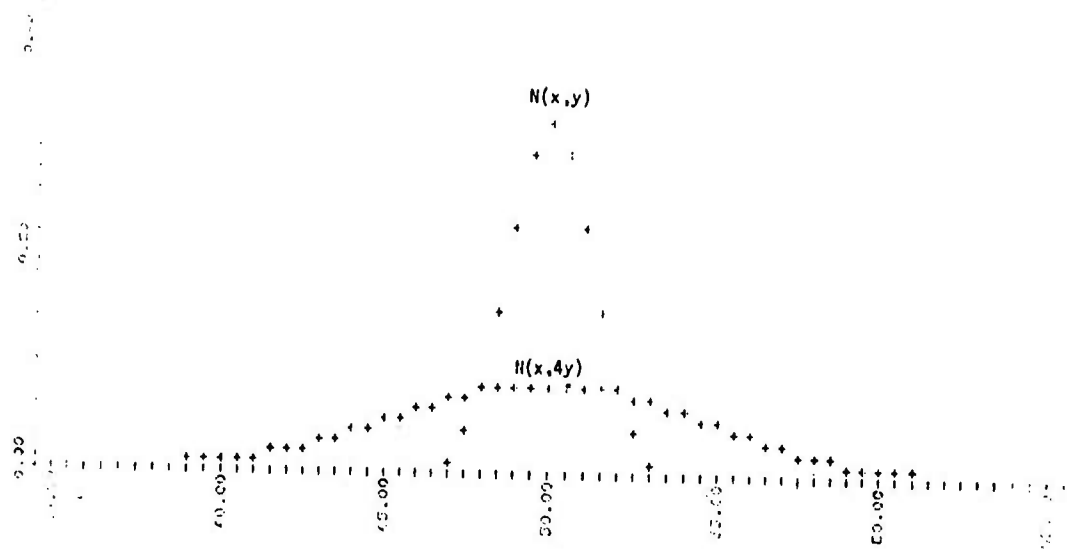H_1: 9 pairs from $G(18,x)$ and 1 pair from $G(10,x)$ - B's

67
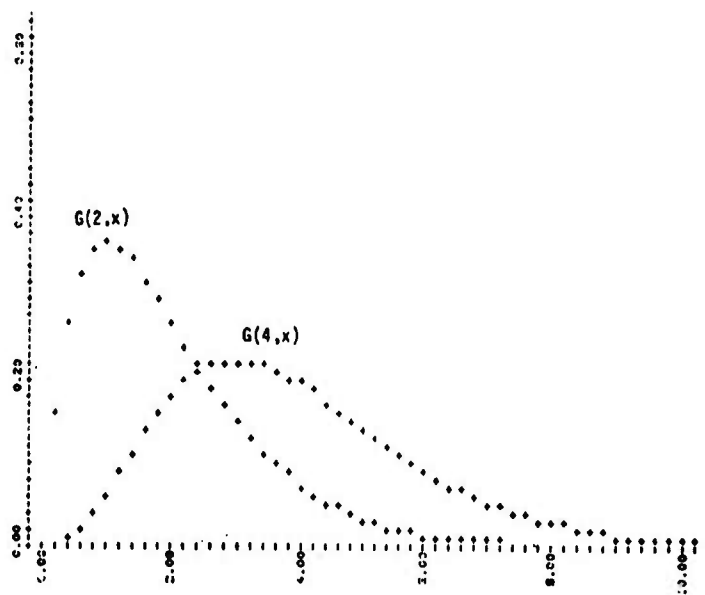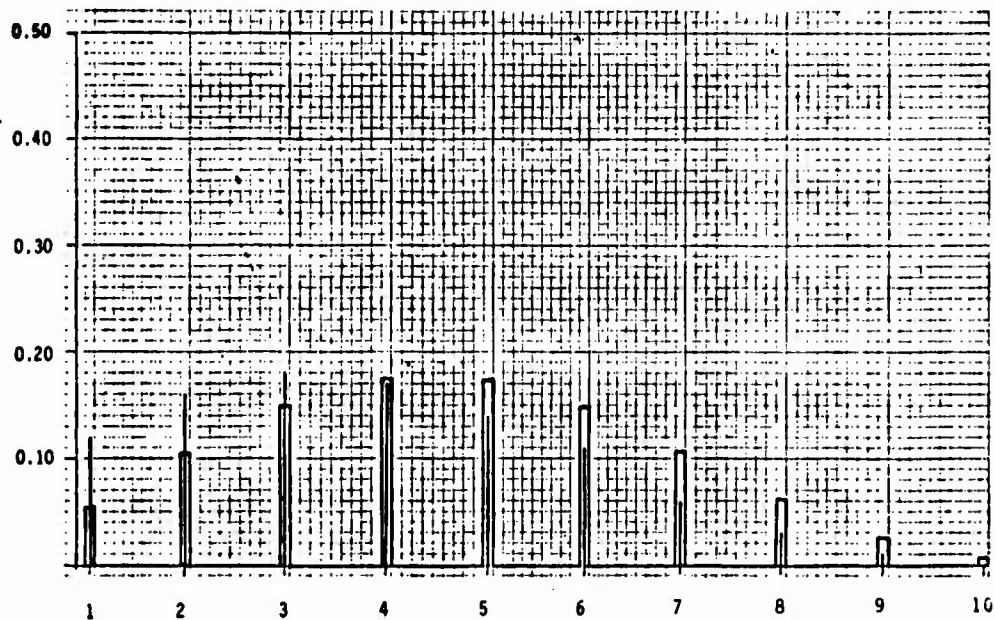
H_0: 10 pairs of observations drawn from the same distribution

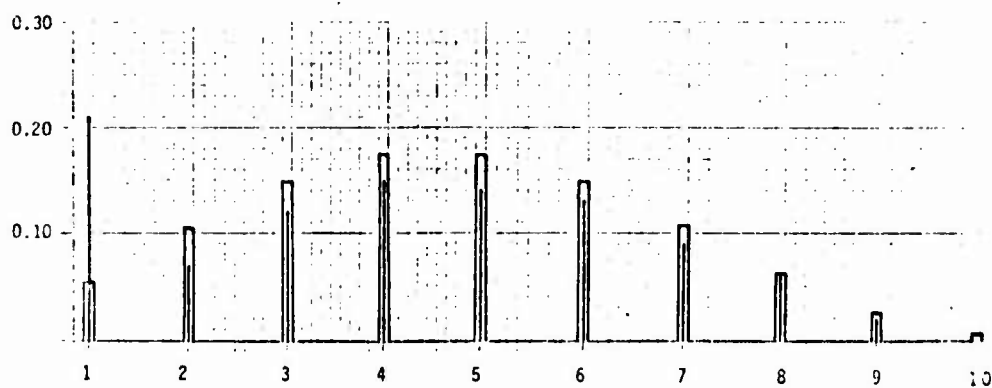H_1: 9 pairs from $G(2,x)$ and 1 pair from $G(4,x)$ - A's
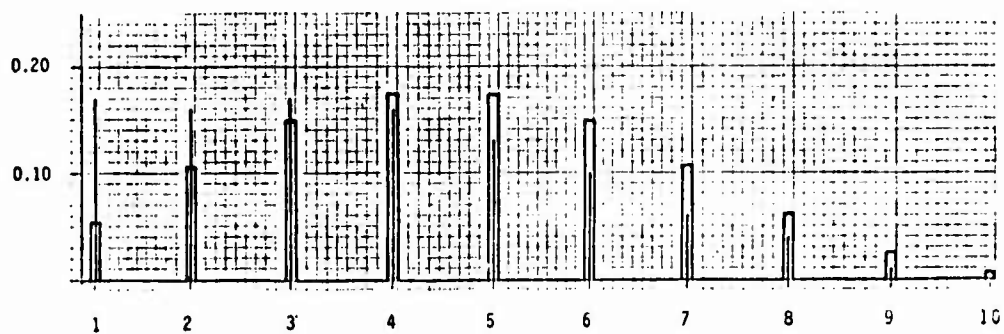
H_0: 10 pairs of observations drawn from the same distribution

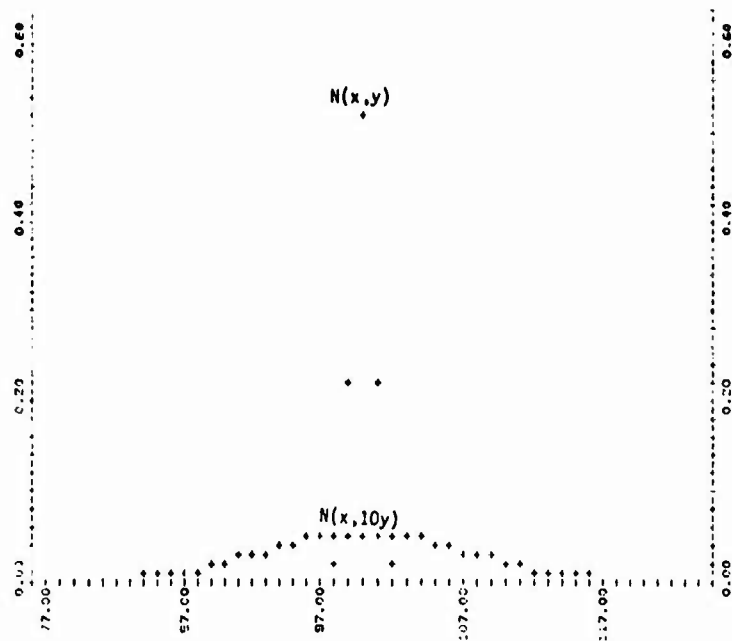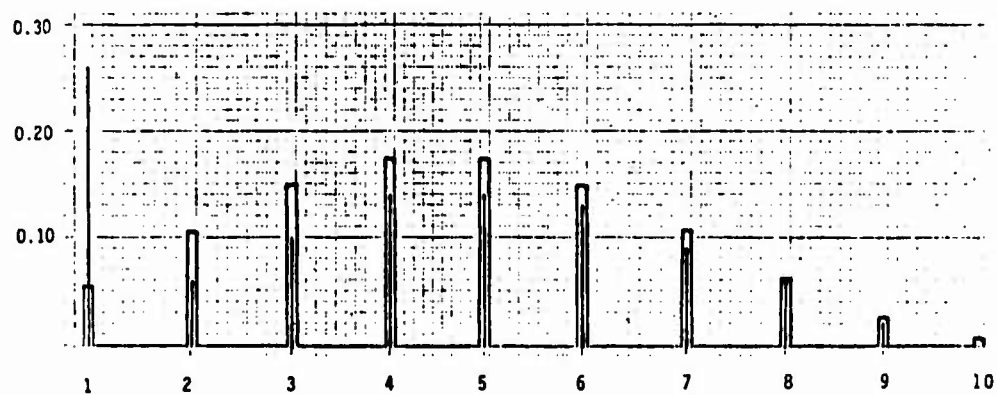H_1: 9 pairs from $G(4,x)$ and 1 pair from $G(2,x)$ - B's
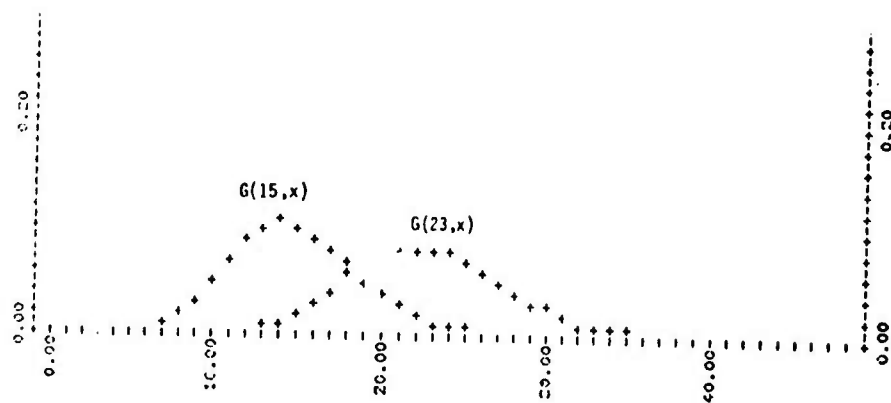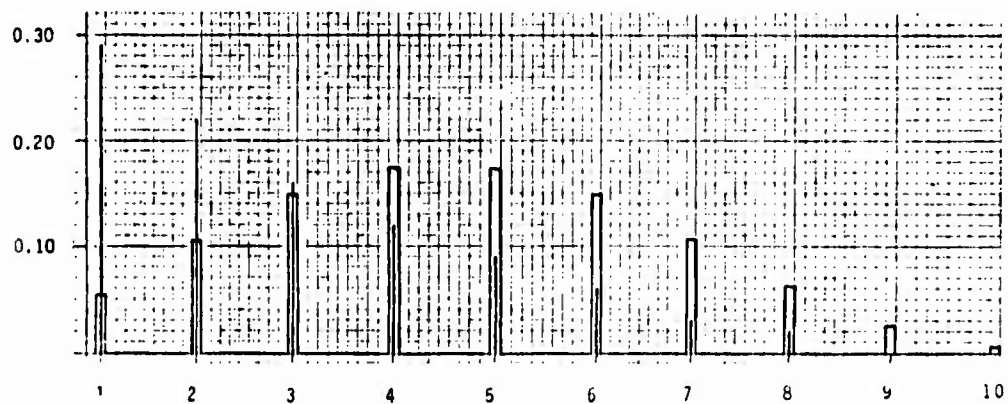
H$_0$: 10 pairs of observations drawn from the same distribution

H$_1$: 9 pairs from G(15,x) and 1 pair from G(23,x) - A's
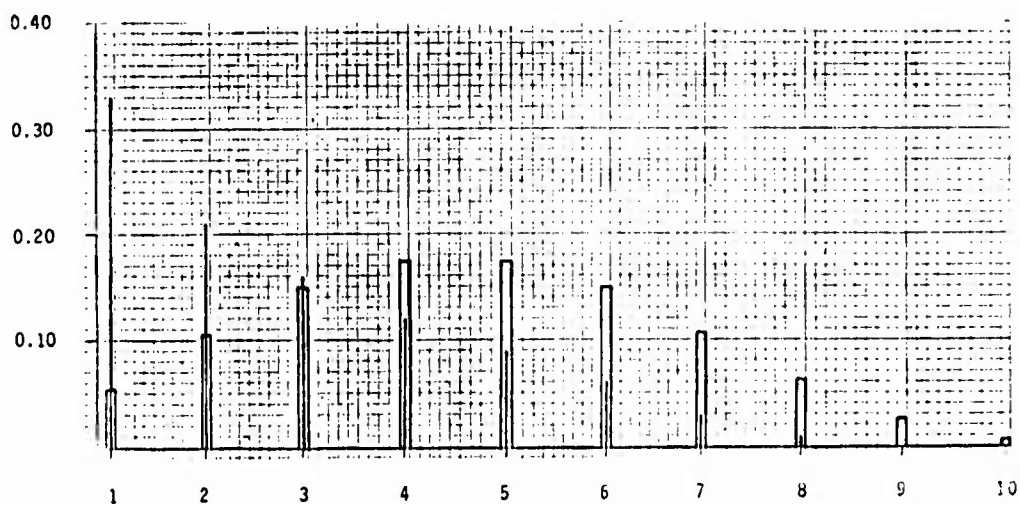
H$_0$: 10 pairs of observations drawn from the same distribution

H$_1$: 9 pairs from G(23,x) and 1 pair from G(15,x) - B's
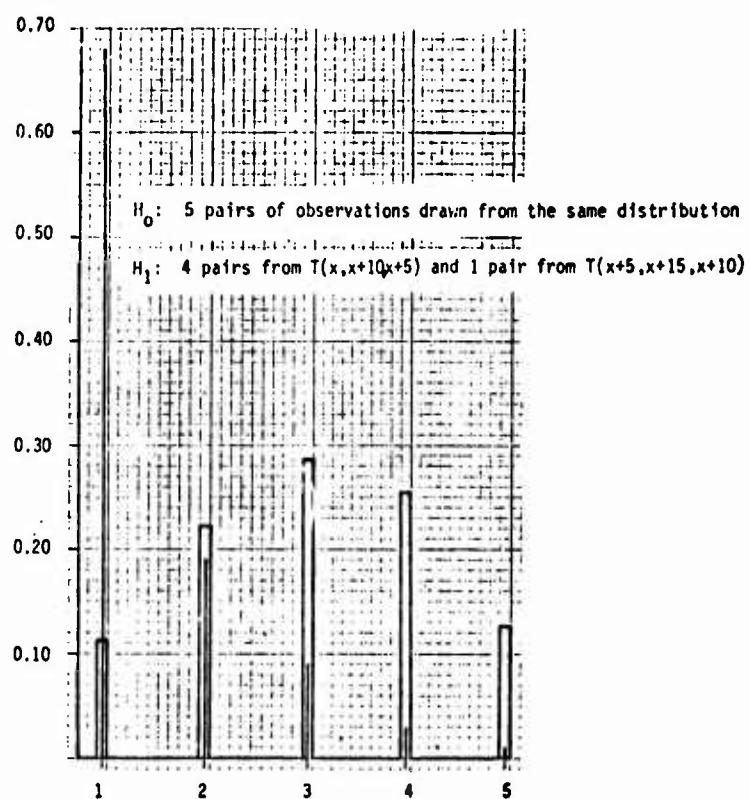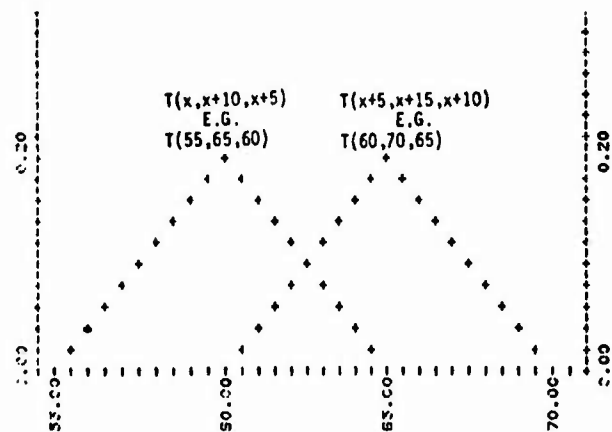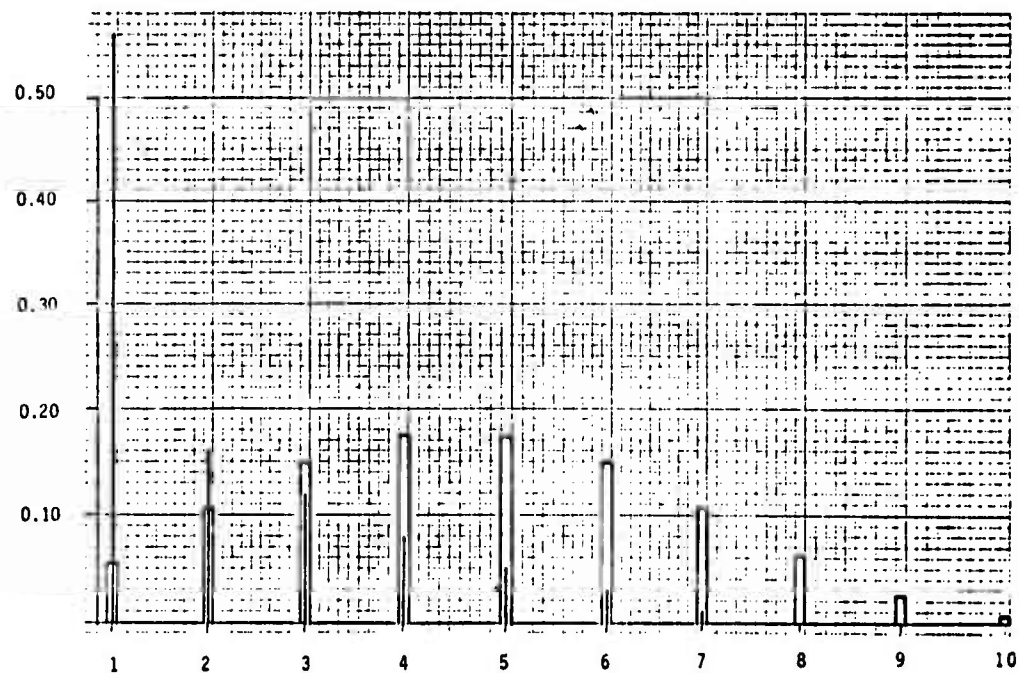
71

T(x,x+10,x+5)   T(x+5,x+15,x+10)
    E.G.   E.G.
  T(55,65,60)   T(60,70,65)

H_o: 5 pairs of observations drawn from the same distribution

H_1:   4 pairs from T(x,x+10,x+5) and 1 pair from T(x+5,x+15,x+10)

H_0: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from $T(x,x+10,x+5)$ and 1 pair from $T(x+5,x+15,X+10)$
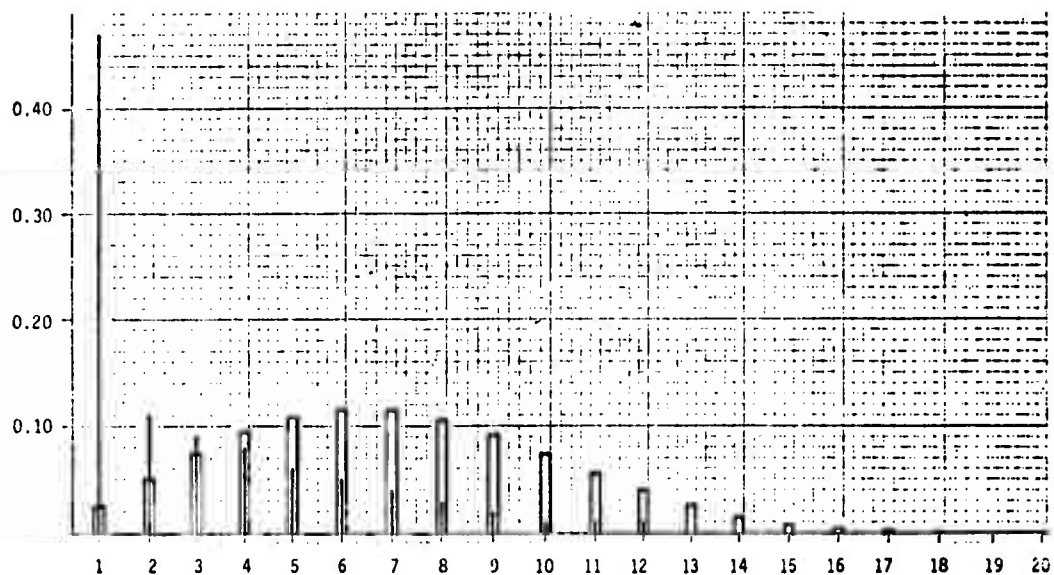
H_0: 20 pairs of observations drawn from the same distribution

H_1: 19 pairs from $T(x,x+10,x+5)$ and 1 pair from $T(x+5,x+15,x+10)$

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H_0: 50 pairs of observations drawn from the same distribution

H_1: 49 pairs from $T(x,x+10,x+5)$ and 1 pair from $T(x+5,x+15, x+10)$

T(x,x+10,x+5)
E.G.
T(56,66,61)

T(x+4,x+14,x+9)
E.G.
T(60,70,65)

0.20   0.20

0.00   0.00

56.00   61.00   66.00
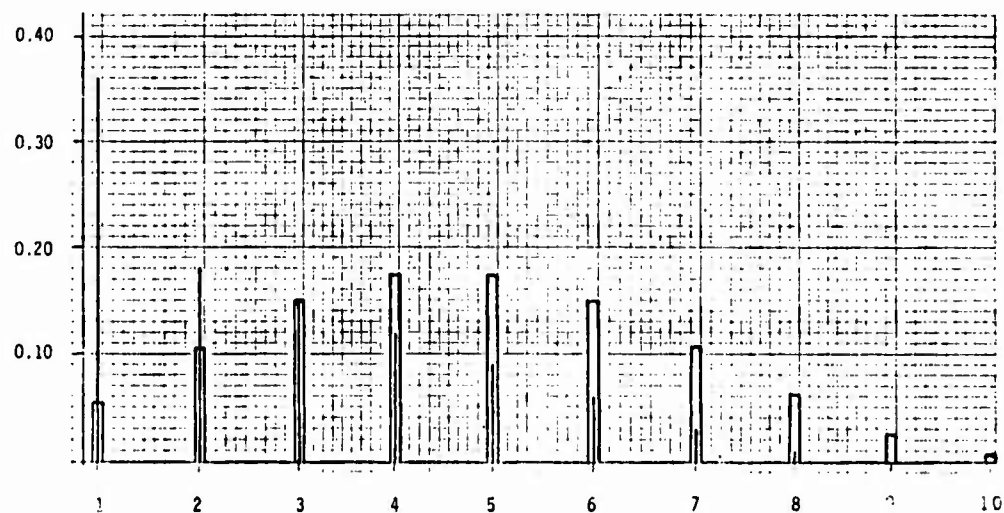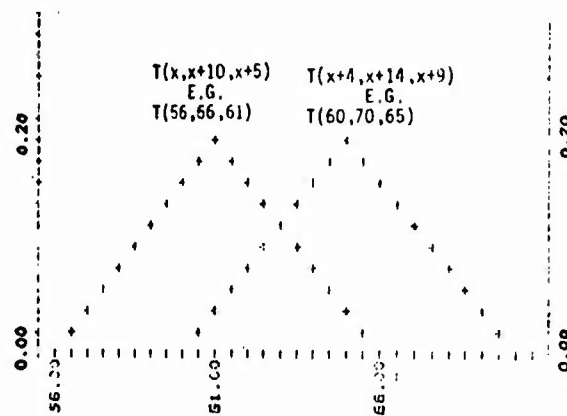
H_0: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from T(x,x+10,x+15) and 1 pair from T(x+4,x+14,x+9)
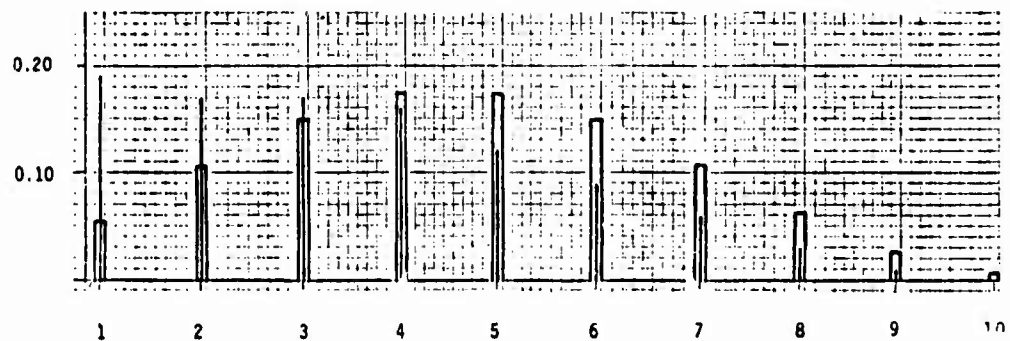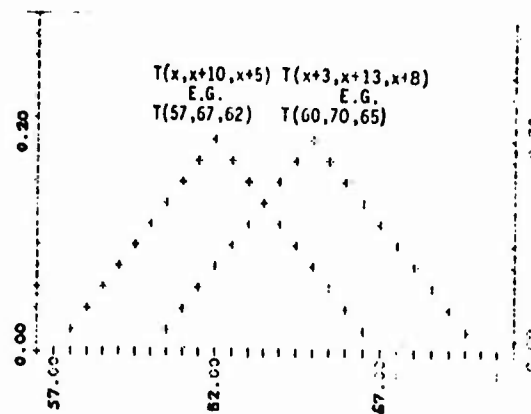
75

T(x,x+10,x+5) T(x+3,x+13,x+8)
E.G. E.G.
T(57,67,62) T(60,70,65)

0.20

0.00

57.00 62.00 67.00

0.20

0.10

1 2 3 4 5 6 7 8 9 10

H_0: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from $T(x,x+10,x+5)$ and 1 pair from $T(x+3,x+13,x+8)$
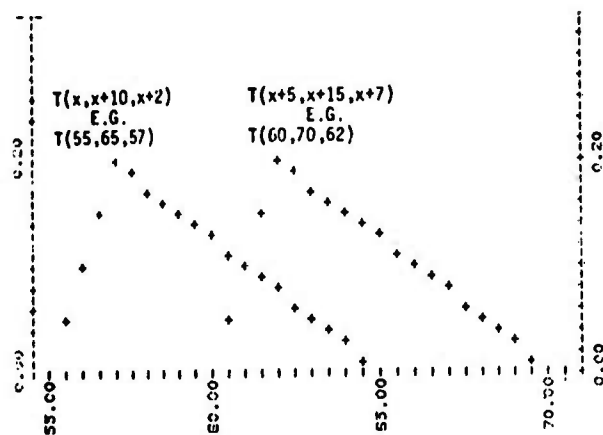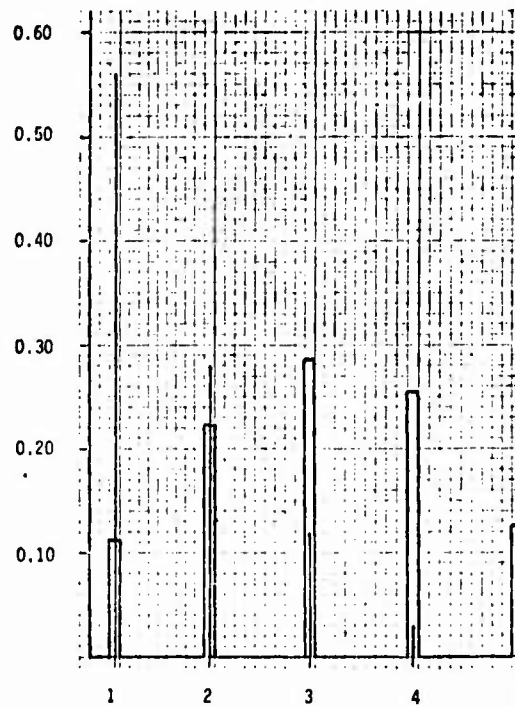
76

H_o: 5 pairs of observations drawn from the same distribution

H_1: 4 pairs from $T(x,x+10,x+2)$ and 1 pair from $T(x+5,x+15,x+7)$ – A's
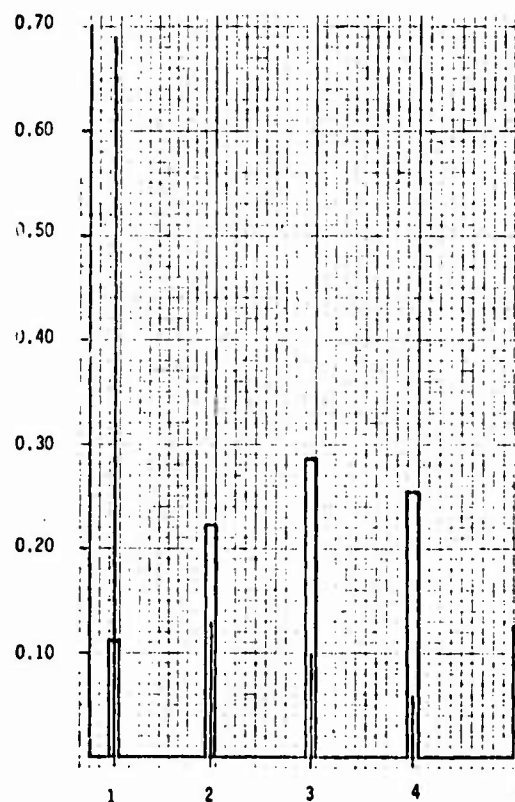
H_o: 5 pairs of observations drawn from the same distribution

H_1: 4 pairs from $T(x+5,x+15,x+7)$ and 1 pair from $T(x,x+10,x+2)$ – B's
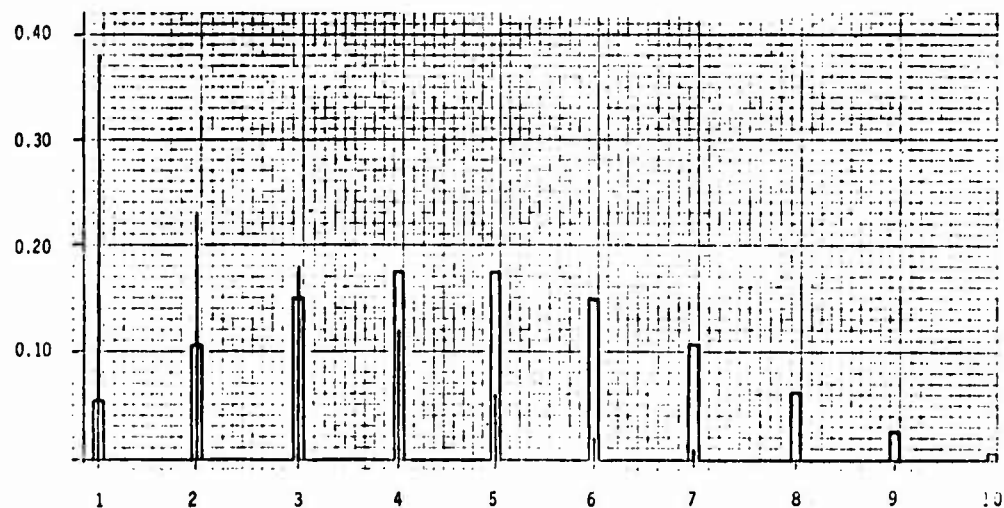
78

H_O: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from $T(x,x+10,x+2)$ and 1 pair from $T(x+5,x+15,x+7)$ - A's
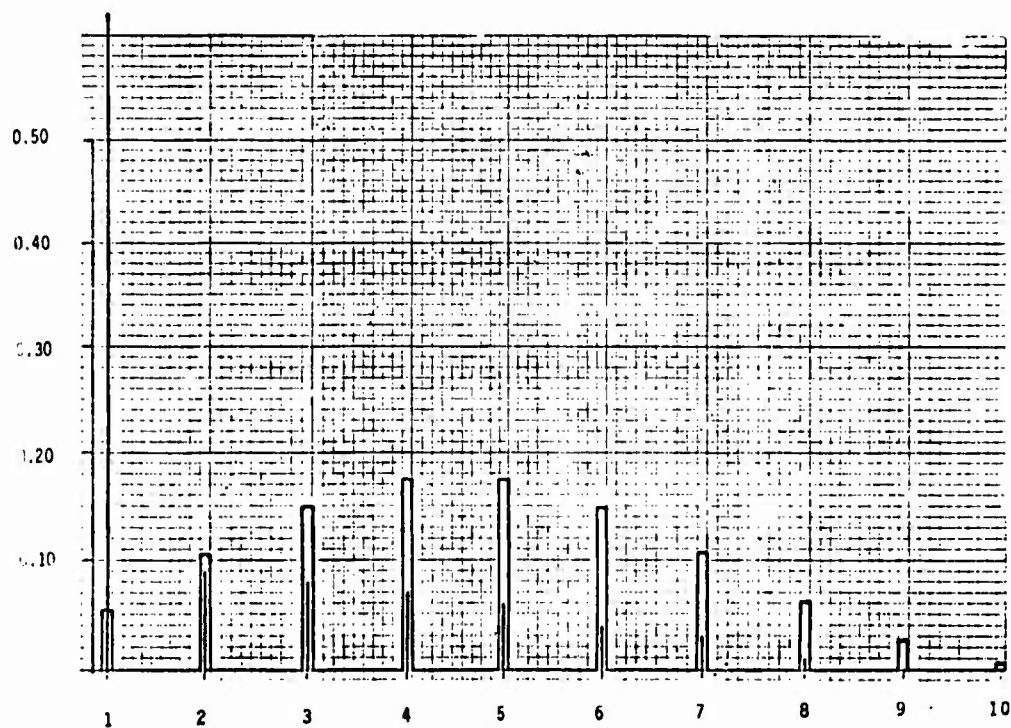
H_O: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from $T(x+5,x+15,x+7)$ and 1 pair from $T(x,x+10,x+2)$ - B's

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B(10,10)

B(2,2)
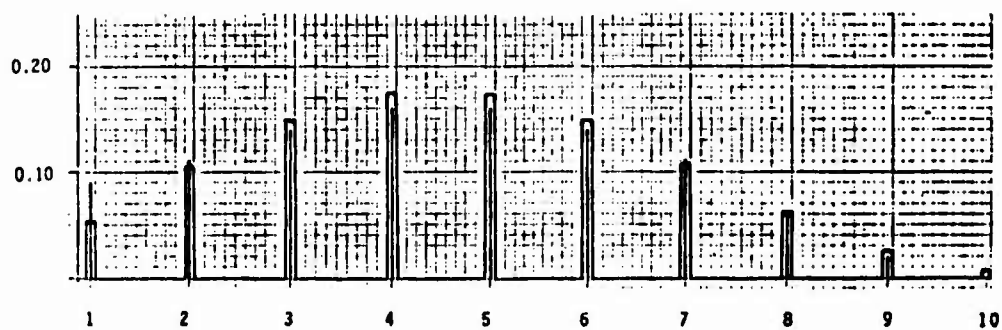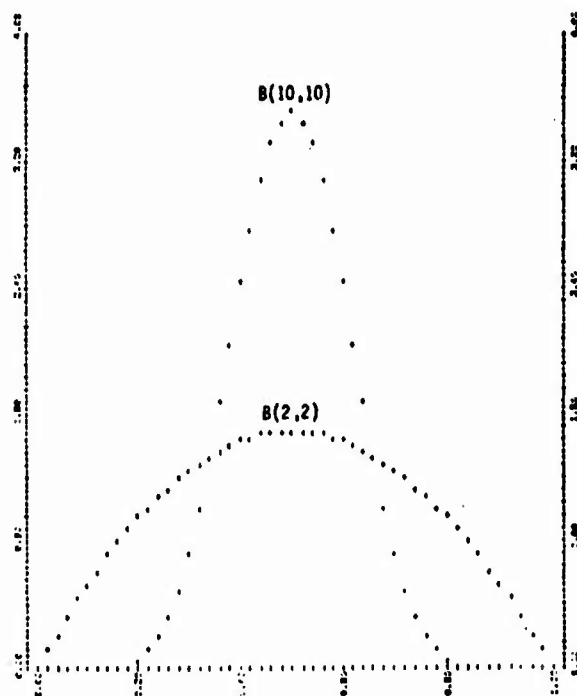
H_0: 10 pairs of observations drawn from the same distribution
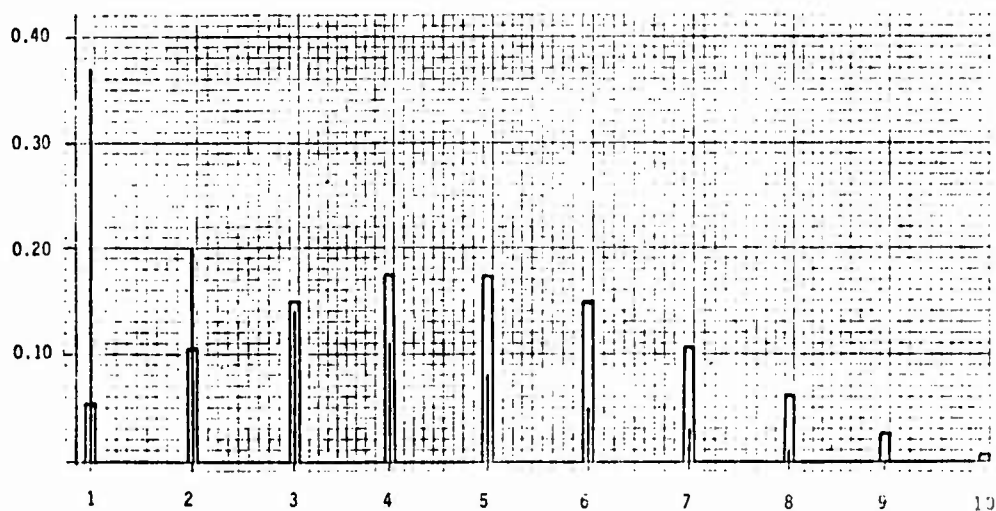
H_1: 9 pairs from B(10,10) and 1 pair from B(2,2)
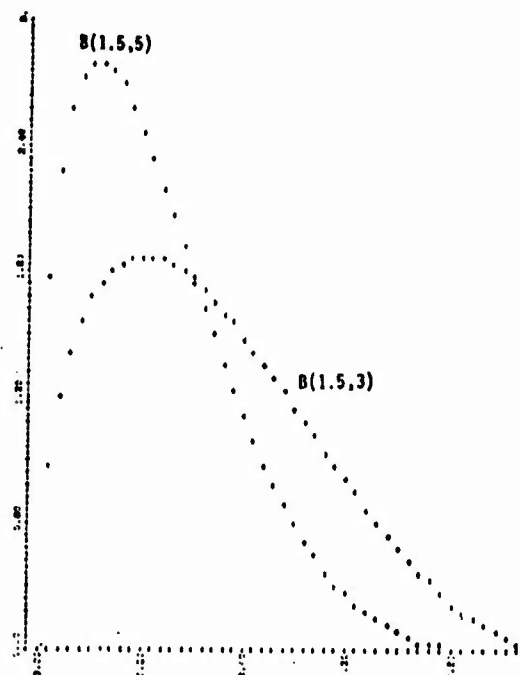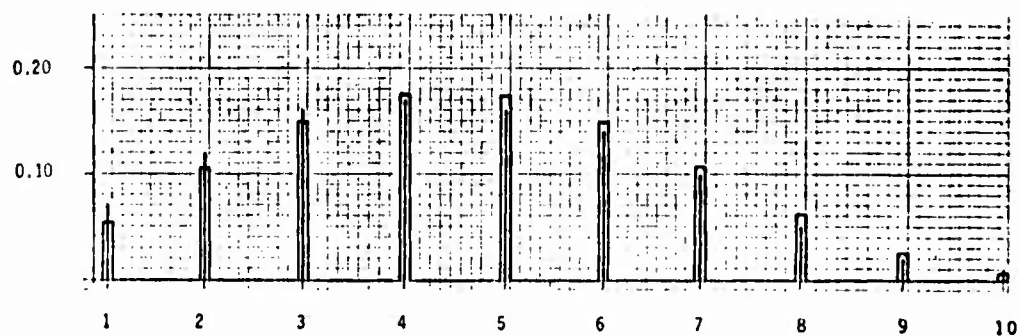
80
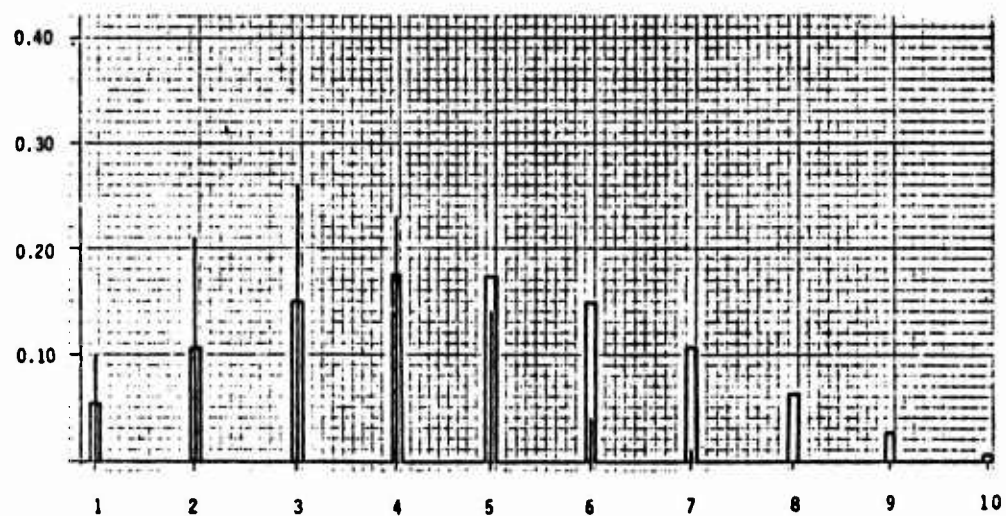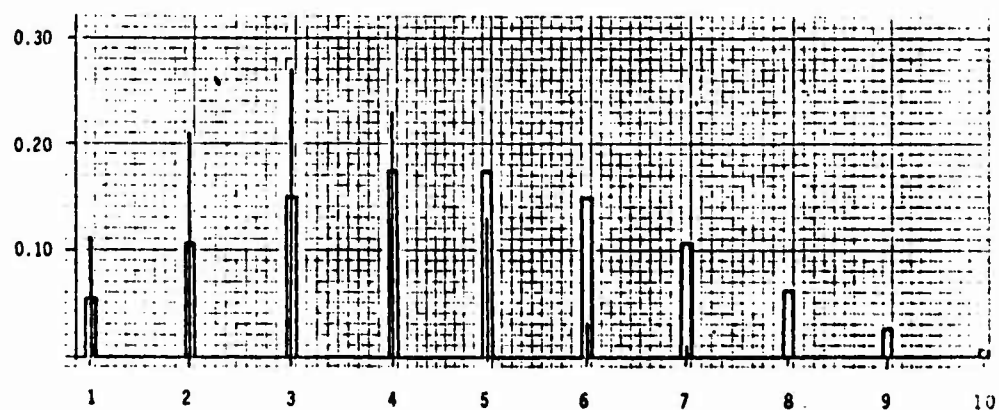
H_0: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from $B(1.5,5)$ and 1 pair from $B(1.5,3)$ - A's
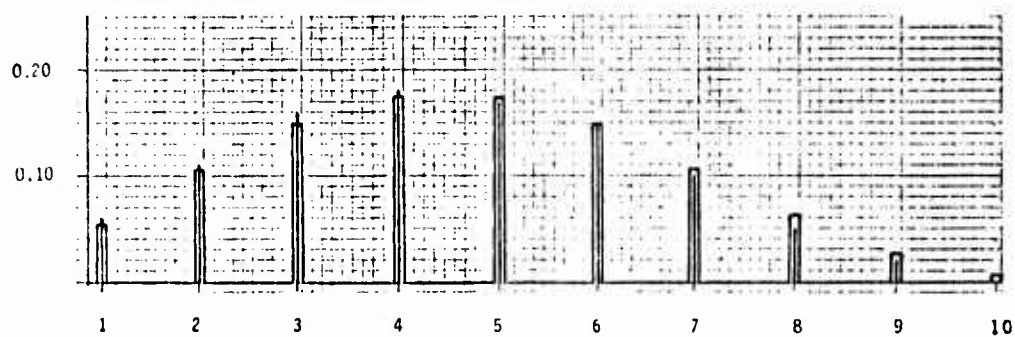
H_0: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from $B(1.5,3)$ and 1 pair from $B(1.5,5)$ - B's
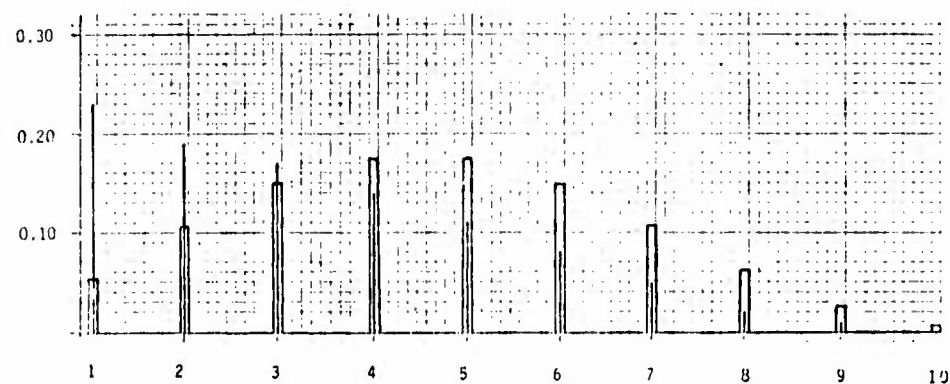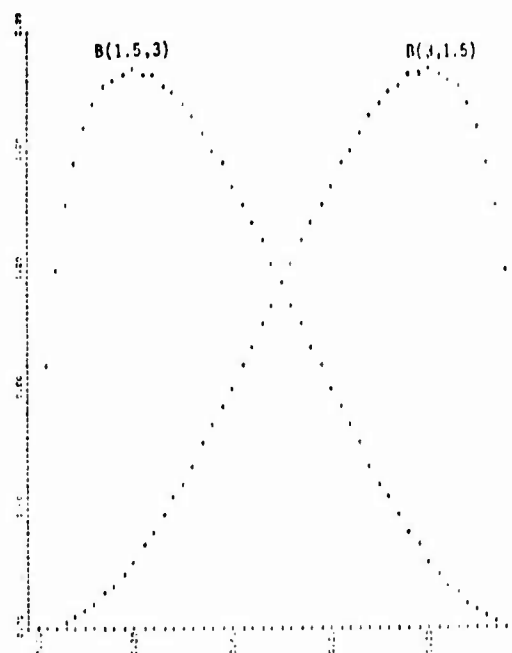
B(1.5,3)   B(3,1.5)

H_0: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from B(1.5,3) and 1 pair from B(3,1.5)
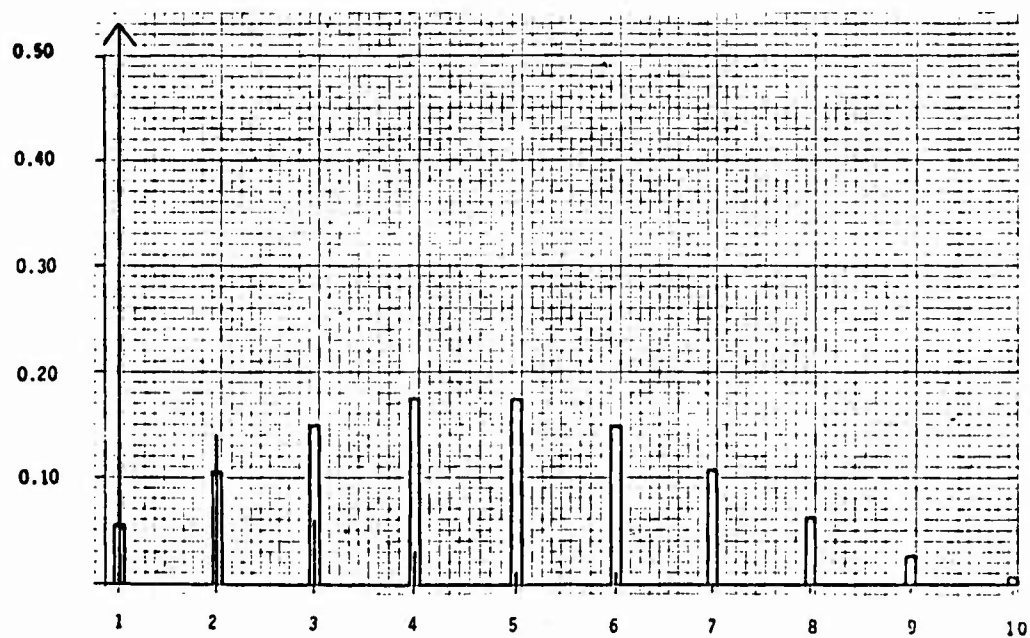
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B(1.5,5)   B(5,1.5)

~ 0.75

H_0: 10 pairs of observations drawn from the same distribution

H_1: 9 pairs from B(1.5,5) and 1 pair from B(5,1.5)

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APPENDIX III

ALIRT

As stated in the introduction to this paper, if "A" is the larger value and "B" is the smaller value of one pair of observations, then the probability for the Length of Initial Run (LIR) of "A"'s or the probability for the LIR of "B"'s can be used to indicate whether data has been drawn from the same distribution or from two different distributions.

In most cases, determining the probability of a LIR equal to N (where N=1,2,3,...) is impractical using analytical methods.

The purpose of this program is to use simulation to estimate the probability of observing a LIR equal to N for an alternative hypothesis that assumes data has been drawn from two different distributions rather than the same distribution (as the null hypothesis assumes).

LIST OF VARIABLES

Variables used in subprogram INITIALIZE

DIST: Value ,determined by input, which specifies the probability distribution.

IREP: Input value for the total number of replications to be performed.

NFILE: Logical unit assignment based on input value.

NAME: Name of output file (when NFILE=8).

DISP: Disposition of output file at termination of program (when NFILE=6).

ITEM: Value (A or B) designating the type of run test to be performed

LETTER: Specifies the type of probability distribution; N-normal, G-gamma, B-beta, or T-triangular.

Variables used in subprogram SORT

SUM: The total number of samples drawn (NSAMP(1)+NSAMP(2)).

ARUN: Length of the initial run of 'A's before the first 'B'.

BRUN: Length of the initial run of 'B's before the first 'A'.

ARESULT(SUM): Array which stores the number of times each 'ARUN' length occurs during the entire simulation.

BRESULT(SUM): Array which stores the number of times each 'BRUN' length occurs during the entire simulation.

Variables used in subprogram REPORT

PERCENT(SUM): Relative frequency destribution of the LIR (ARESULT(SUM)/IREP; BRESULT(SUM)/IREP).

CUM(SUM): Cumulative frequency distribution of the LIR.

ANSWER: Input value which determines the status of bar graph output of frequency distribution (print:YES or NO).

Variables used in subprogram NORMAL

MU(1): Mean for distribution 1.

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SIGMA(I): Standard deviation for distribution I.

X: An independent identically distributed uniform
random number generated by the VAX 11/780 subprogram 'RAN'.

T= (2X-1): A test value to determine acceptability of the
generated random numbers.

Y: $N^{\sim}(0,1)$.

X1,X2: $N^{\sim}(MU,SIGMA)$'s.

Variables used in subprogram GAMMA

ALPHA(I) I=1,2: Shape parameter.

BETA(I) I=1,2: Scale parameter.

R: An independent identically distributed uniform
random number generated by the VAX 11/780 subprogram 'RAN'.

TEST & W: Test values used to determine acceptability
of the generated random numbers.

Y: $GAM^{\sim}(ALPHA(I),1)$ I=1,2.

X(I) I=1,2: $GAM^{\sim}(ALPHA(I),BETA(I))$.

Variables used in subprogram BETA

ALPHA(I,J) I,J=1,2: Shape parameters.

R: An independent identically distributed uniform
random number generated by the VAX 11/780 subprogram 'RAN'.

TEST & W: Test values used to determine acceptability
of the generated random numbers.

Y: $GAM^{\sim}(ALPHA(I),1)$ I=1,2.

X1(I) I=1,2: $GAM^{\sim}(ALPHA(I),1)$ I=1,2.

X2(I) I=1,2: $BET^{\sim}(ALPHA(I),ALPHA(J))$ I,J=1,2.

Variables used in subprogram TRIANG

G: Minimum value. Location parameter.

H: Maximum value. Max-Min=Scale parameter.

C: Mode. Shape parameter.

R: An independent identically distributed uniform
random number generated by the VAX 11/780 subprogram 'RAN'.

X2,X3:   TRIANG~(0,1,(mode-min)/(max-min)).

X(I) I=1,2:   TRIANG~(min,max,mode).

    Variables common to subprograms

NSAMP(I) I=1,2:   The number of two-observation samples
    required for distribution I.

A(SUM) SUM=1,NSAMP(1)+NSAMP(2):   Array of greater values of
    each two-observation sample from the combined NSAMP'S.

B(SUM) SUM=1,NSAMP(1)+NSAMP(2):   Array of lesser values of
    each two-observation sample from the combined NSAMP'S.

COUNT: Counter for current number of replications; simula-
    tion terminates when COUNT=IREP.

II: Initial value required for random number generation.

PROGRAM LIMITATIONS

Subprogram TRIANG: The Mode 'C' used to generate the
    TRIANG(0,1,C), where C=(mode-min)/(max-min), is restricted
    to 0<C<1.

Subprogram GAMMA: ALPHA>1.

Subprogram BETA: ALPHA(I,J)>1 I=1,2.

Input distributions must both be of the same type.

Input distribution type is limited to normal, gamma, beta,
    or triangular.

The number of replications must be less than or equal to 100,000.

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START

CALL INITIALIZE

CALL APPROPRIATE
DISTRIBUTION SUBROUTINE

a

OUTPUT SIMULATION
RESULTS AND GRAPHS

STOP

START

SELECT OUTPUT
FILE VARIABLES

SELECT SPECIFIC
LIR TEST(A OR B)

SELECT SPECIFIC
PROBABILITY DISTRIBUTION

SELECT NO. OF
REPLICATIONS

RETURN

LETTER=N
?   yes   1a
no

LETTER=G
?   yes   1b
no

LETTER=B
?   yes   1c
no

1d

START

PRINT HEADING OF
OUTPUT FILE

CALCULATE RELATIVE
AND CUMULATIVE FREQUENCY

CALL SUBROUTINE
GRAPH

RETURN

START

PRINT BAR
GRAPH OF THE REL-
ATIVE AND CUMULATIVE
FREQUENCY DIST-
RIBUTIONS
                      no
           yes

PRINT BAR GRAPHS

RETURN

T REPORTS LIS

```
        1b ──────▶ START

        SELECT ALPHA,BETA, AND
           NO. OF SAMPLES FOR
        THE FIRST DISTRIBUTION

        SELECT ALPHA,BETA, AND
           NO. OF SAMPLES FOR
        THE SECOND DISTRIBUTION

           GENERATE TWO
              URN'S

           USE ONE URN TO
        GENERATE A G~(α,1),
           USE BOTH URN'S TO
        TEST FOR NEXT STEP

              URN'S              no
            ACCEPTABLE  ──────────────▶
               ?
              yes

            USE G~(α,1) TO
        GENERATE A G(α,θ)

             HAVE TWO           no
           G~(α,θ) BEEN  ──────────────▶
            GENERATED
               ?
              yes

        ASSIGN LARGER NO.
        TO 'A' ARRAY AND
        SMALLER NO. TO 'B'
            ARRAY

              TOTAL             no
         NUMBER OF SAMPLES  ─────────────▶
        GENERATED FOR EACH
           DISTRIBUTION
               ?
              yes

           CALL SORT

              COUNT          yes
              IREP   ──────────────▶
               ?
              no

           RETURN ──────▶ 4
```

START

SELECT ALPHA1,ALPHA2,AND
NO. OF SAMPLES FOR
THE FIRST DISTRIBUTION

SELECT ALPHA1,ALPHA2,AND
NO. OF SAMPLES FOR
THE SECOND DISTRIBUTION

GENERATE A $G \sim (\alpha_1, 1)$
AND A $G \sim (\alpha_2, 1)$

USE THE PAIR OF
GAMMAS TO GENERATE
A $B \sim (\alpha_1, \alpha_2)$

ARE TWO
BETAS GENERATED
?

no

yes

ASSIGN LARGER NO.
TO 'A' ARRAY AND
SMALLER NO. TO 'B'
ARRAY

TOTAL
NUMBER OF SAMPLES
GENERATED FOR EACH
DISTRIBUTION
?

no

yes

CALL SORT

COUNT <
IREP
?

yes

no

2

RETURN

4

93

```
                          ┌──◄── START
                          1

                    ┌─────────────────┐
                    │ SELECT A,B,C,AND │
                    │ NO. OF SAMPLES FOR│
                    │THE FIRST DISTRIBUTION│
                    └─────────────────┘
                             │
                    ┌─────────────────┐
                    │ SELECT A,B,C,AND │
                    │ NO. OF SAMPLES FOR│
                    │THE SECOND DISTRIBUTION│
                    └─────────────────┘
                             │
                    ┌─────────────┐
                    │ GENERATE A  │
                    │ URN         │
                    └─────────────┘
                             │
                    ┌─────────────┐
                    │ USE URN TO  │
                    │ GENERATE A  │
                    │ T''(a,b,c)  │
                    └─────────────┘
                             │
                         ╱ TWO ╲      no
                       ╱ T''(a,b,c) GEN ╲─────►
                       ╲ ERATED ╱
                         ╲  ?  ╱
                             │ yes
                    ┌─────────────┐
                    │ ASSIGN LARGER│
                    │ NO. TO 'A'  │
                    │ ARRAY AND   │
                    │ SMALLER NO. TO│
                    │ 'B' ARRAY   │
                    └─────────────┘
                             │
                      ╱ TOTAL ╲     no
                    ╱ NUMBER OF ╲
                    ╱ SAMPLES GEN- ╲──────►
                    ╲ ERATED FOR EACH╱
                     ╲ DISTRIBUTION ╱
                       ╲    ?   ╱
                           │ yes
                     ◄── CALL SORT
                         3
                           │
                       ╱ COUNT: ╲    yes
                       ╲ IREP  ╱─────►
                        ╲  ?  ╱
                           │ no
                        RETURN
                           │
                           4
```

```
*********************MAIN PROGRAM***********************************

        COMMON/ALG/A(50),B(50),SUM
        COMMON/ALL/NFILE,IREP,ITEM
        COMMON/OUT/ARESULT(50),BRESULT(50),PERCENT(50)
        COMMON/FILE/NAME,DISP
        CALL INITIALIZE(DIST)
 19     GOTO(1,2,3,4),DIST
  1     CALL NORMAL
        GOTO 10
  2     CALL GAMMA
        GOTO 10
  3     CALL BETA
        GOTO 10
  4     CALL TRIANG
 10     CALL REPORT
        STOP
        END



****************************************************************
        *                                                *
        *   This subroutine inputs the type of probability *
        *   distribution and the number of replications.   *
        *                                                *
****************************************************************


        SUBROUTINE INITIALIZE(DIST)
        COMMON/ALL/NFILE,IREP,ITEM
        COMMON/FILE/NAME,DISP
        CHARACTER*4 LETTER,CHOICE
        CHARACTER*13 NAME,DISP
 35     WRITE(6,31)
 31     FORMAT(T2,'Output may be written to a file and then printed ',
     /'or printed at this terminal.',/,T2,'Enter "F" for file or "T"',
     /  ' for terminal.')
        READ(5,79)CHOICE
        IF(CHOICE.EQ.'F')THEN
         NFILE=8
        ELSE IF(CHOICE.EQ.'T')THEN
         NFILE=6
         GOTO 3
        ELSE
         GOTO 35
        END IF
        WRITE(6,12)
 12     FORMAT(T2,'Enter the name of the output file.')
        READ(5,17)NAME
 17     FORMAT(A)
        WRITE(6,21)
 21     FORMAT(T2,'Select the disposition of the output file.',
     /  /,T2,'Type: KEEP,PRINT,or PRINT/DELETE.')
        READ(6,27)DISP
 27     FORMAT(A)
        OPEN(8,FILE=NAME,STATUS='NEW',DISP=DISP)
  3     WRITE(6,72)
 72     FORMAT(T2,'Which type of test is to be performed?',/,T2,
     /  'Enter "ARUN" if you wish to know the length of the A  ',
     /  'run before the first B.',/,' Enter "BRUN" if you wish the',
```

```
/     ' length of the B run before the first A.',/,T2,'Enter "BOTH"
/     ' if you would like the results of both tests.')
      READ(5,79)TEST
79    FORMAT(A)
      IF(TEST.EQ.'ARUN')THEN
       ITEM=1
      ELSE IF(TEST.EQ.'BRUN')THEN
       ITEM=2
      ELSE IF(TEST.EQ.'BOTH')THEN
       ITEM=3
      ELSE
       GOTO 3
      END IF
13    WRITE(6,10)
10    FORMAT(12,'Select the type of distribution you would ',
   /    'like.',/,' Type the first letter of the name to make ',
   /    'your input.')
      READ(5,23)LETTER
23    FORMAT(A)
      IF(LETTER.EQ.'N')THEN
       DIST=1
      ELSE IF(LETTER.EQ.'G')THEN
       DIST=2
      ELSE IF(LETTER.EQ.'B')THEN
       DIST=3
      ELSE IF(LETTER.EQ.'T')THEN
       DIST=4
      ELSE
       GOTO 13
      END IF
92    WRITE(6,93)
93    FORMAT(T2,'Enter the number of replications ',
   /    'to be performed.')
91    READ(5,*)IREP
      IF(IREP.GT.100000)GOTO 92
      RETURN
      END


************************************************************************
      *                                                    *
      *      This subroutine generates random numbers      *
      *             from the normal distribution.          *
      *                                                    *
************************************************************************

      SUBROUTINE NORMAL
      DIMENSION V(2),MU(2),SIGMA(2),NSAMP(2)
      COMMON/ALG/A(50),B(50),SUM
      COMMON/ALL/BFILE,IREP,ITEM
      COMMON/FILE/NAME,DISP
      CHARACTER*13 NAME,DISP
      REAL MU
      INTEGER COUNT,I1,SUM
      COUNT=0
      IF( FILE.EQ.6)GOTO 6
      OPEN(6,FILE=NAME,STATUS='OLD',DISP=DISP)
6     PRINT 5
5     FORMAT(1X,'Enter mu, sigma and number of samples',
   /    ' for the',/,T2,
   /    'first normal distribution separated by commas.')
```
97

```
      READ(5,*)MU(1),SIGMA(1),NSAMP(1)
      WRITE(6,17)
17    FORMAT(T2,'Enter mu, sigma and number of samples',
     /  ' for the',/,T2,
     /  'second normal distribution separated by commas.')
      READ(5,*)MU(2),SIGMA(2),NSAMP(2)
      WRITE(6,19)
19    FORMAT(T2,'Enter a five-digit number for random',
     /  ' number generation.')
      READ(5,*)I1
      WRITE(NFILE,98)MU(1),SIGMA(1),MU(2),SIGMA(2),NSAMP(1),
     /  NSAMP(2),I1
98    FORMAT(T2,'Input distributions are N~(',F5.2,',',F5.2,') and',
     /  ' N~(',F5.2,',',F5.2,').',/,T2,'Number of samples from each ',
     /  'distribution is ',I2,' and ',
     /  I2,' respectively.',/,T2,'The random number seed for this ',
     /  'run is ',I6,'.')
97    COUNT=COUNT+1
      J=0
      SUM=0
100   J=J+1
      N=0
90    T=0
      DO 80 I=1,2
      X=RAN(I1)
      V(I)=2.*X-1.
      W=V(I)*V(I)
      T=T+W
80    CONTINUE
      IF(T.GT.1.)GO TO 90
      Y1=LOG(T)
      Y2=-2.*Y1/T
      Y=SQRT(Y2)
      N=N+1
      SUM=SUM+1
      X1=MU(J)+SIGMA(J)*V(1)*Y
      X2=MU(J)+SIGMA(J)*V(2)*Y
      IF(X1.GT.X2)THEN
      A(SUM)=X1
      B(SUM)=X2
      ELSE
      A(SUM)=X2
      B(SUM)=X1
      ENDIF
      IF(N.LT.NSAMP(J))GO TO 90
      IF(J.LT.2)GO TO 100
      CALL SORT
      IF(COUNT.LT.IREP)GOTO 97
      RETURN
      END
```

```
****************************************************************
*                                                              *
*        This subroutine generates random numbers              *
*                from the gamma distribution.                  *
*                                                              *
****************************************************************
```

```
      SUBROUTINE GAMMA
      DIMENSION U(2),X(2),ALPHA(2),BETA(2),NSAMP(2)
```

98

```
          COMMON/ALG/A(50),B(50),SUM
          COMMON/ALL/MFILE,IREP,ITER
          COMMON/FILE/NAME,DISP
          CHARACTER*13 NAME,DISP
          INTEGER COUNT,I1,SUM
          REAL LTHETA,LNZ
          COUNT=0
          IF(MFILE.EQ.6)GOTO 6
          OPEN(8,FILE=NAME,STATUS='OLD',DISP=DISP)
    6     PRINT 10
   10     FORMAT(1X,'Enter alpha, beta and number of samples for',
      /   ' the',/,12,
      /   'first gamma distribution separated by commas.')
          READ(5,*)ALPHA(1),BETA(1),NSAMP(1)
          IF(ALPHA(1).LE.1)GOTO 6
    8     PRINT 11
   11     FORMAT(1X,'Enter alpha, beta and number of samples for',
      /   ' the',/,12,
      /   'second gamma distribution separated by commas.')
          READ(5,*)ALPHA(2),BETA(2),NSAMP(2)
          IF(ALPHA(2).LE.1)GOTO 6
          PRINT 12
   12     FORMAT(12,'Enter a five-digit number for random',
      /   ' number generation.')
          READ(5,*)I1
          WRITE(MFILE,96)ALPHA(1),BETA(1),ALPHA(2),BETA(2),NSAMP(1),
      /   NSAMP(2),I1
   96     FORMAT(12,'Input distributions are G~(',F5.2,',',F5.2,') and',
      /   ' G~(',F5.2,',',F5.2,').',/,T2,'Number of samples from ',
      /   'each distribution is ',12,' and ',
      /   12,' respectively.',/,T2,'The random number seed for this ',
      /   'run is ',I6,'.')
          THETA=4.5
          LTHETA=LOG(4.5)
          D=1.+LTHETA
          FOUR=LOG(4.)
   97     COUNT=COUNT+1
          SUM=0
          K=0
   60     K=K+1
          G1=2.*ALPHA(K)-1.
          G2=SQRT(G1)
          G=1./G2
          Q=ALPHA(K)+G2
          H=ALPHA(K)-FOUR
          M=0
   80     J=0
   20     J=J+1
   50     DO 30 I=1,2
          R=RAN(I1)
          U(I)=R
   30     CONTINUE
          V1=U(1)/(1.-U(1))
          V2=LOG(V1)
          V=G*V2
          Y1=EXP(V)
          Y=ALPHA(K)*Y1
          Z=U(2)*U(1)*U(1)
          W=H+Q*V-Y
          TEST=W+D-LTHETA*Z
          IF(TEST.GE.0.)GO TO 40
```

99

```
          LNZ=LOG(Z)
          IF(W.GE.LNZ)GO TO 40
          GO TO 50
   40     X(J)=Y*BETA(K)
          IF(J.LT.2)GO TO 20
          SUM=SUM+1
          N=N+1
          IF(X(1).LT.X(2))THEN
             A(SUM)=X(2)
             B(SUM)=X(1)
          ELSE
             A(SUM)=X(1)
             B(SUM)=X(2)
          END IF
          IF(N.LT.NSAMP(K))GO TO 80
          IF(K.LT.2)GO TO 60
          CALL SORT
          IF(COUNT.LT.IREP)GOTO 97
          RETURN
          END


  ***********************************************************************
              *                                                 *
              *      This subroutine generates random numbers   *
              *           from the beta distribution.           *
              *                                                 *
  ***********************************************************************


          SUBROUTINE BETA
          DIMENSION X1(2),X2(2),U(2),ALPHA(2,2),NSAMP(2)
          COMMON/ALG/A(50),B(50),SUM
          COMMON/ALL/NFILE,IREP,ITEM
          COMMON/FILE/NAME,DISP
          CHARACTER*13 NAME,DISP
          REAL LTHETA,LNZ
          INTEGER COUNT,I1,N,O,SUM
          COUNT=0
          IF(NFILE.EQ.6)GOTO 3
          OPEN(8,FILE=NAME,STATUS='OLD',DISP=DISP)
   3      PRINT 10
   10     FORMAT(1X,'Enter alpha1, alpha2 and number of samples ',
         /  'for the',/,T2,
         /  'first beta distribution separated by commas.')
          READ(5,*)ALPHA(1,1),ALPHA(2,1),NSAMP(1)
          PRINT 11
   11     FORMAT(1X,'Enter alpha1, alpha2 and number of samples ',
         /  'for the',/,T2,
         /  'second beta distribution separated by commas.')
          READ(5,*)ALPHA(1,2),ALPHA(2,2),NSAMP(2)
          PRINT 12
   12     FORMAT(T2,'Enter a five-digit number for random',
         /  ' number generation.')
          READ(5,*)I1
          WRITE(NFILE,98)ALPHA(1,1),ALPHA(2,1),ALPHA(1,2),ALPHA(2,2),
         /  NSAMP(1),NSAMP(2),I1
   98     FORMAT(T2,'Input distributions are B~(',F5.2,',',F5.2,') and',
         /  ' B~(',F5.2,',',F5.2,').',/,/,T2,'Number of samples from',
         /  ' each distribution is ',I2,' and ',
         /  I2,' respectively.',/,T2,'The random number seed for this ',
```

100

```fortran
     /    'run is ',I6,'.')
          THETA=4.5
          LTHETA=LOG(4.5)
          T=1.+LTHETA
          FOUR=LOG(4.)
97        COUNT=COUNT+1
          G=0
          SUM=0
          DO 90 N=1,2
          G=G+1
          DO 80 M=1,NSAMP(N)
          SUM=SUM+1
          DO 70 L=1,2
          DO 60 K=1,2
          G1=2.*ALPHA(K,N)-1.
          G2=SQRT(G1)
          G=1./G2
          G=ALPHA(K,N)+G2
          H=ALPHA(K,N)-FOUR
50        DO 30 I=1,2
          R=RAN(I1)
          U(I)=R
30        CONTINUE
          V1=U(1)/(1.-U(1))
          V2=LOG(V1)
          V=G*V2
          Y1=EXP(V)
          Y=ALPHA(K,N)*Y1
          Z=U(2)*U(1)*U(1)
          W=H+G*V-Y
          TEST=W+D-THETA*Z
          IF(TEST.GE.0.)GO TO 40
          LNZ=LOG(Z)
          IF(W.GE.LNZ)GO TO 40
          GO TO 50
40        X1(K)=Y
60        CONTINUE
          X2(L)=X1(1)/(X1(1)+X1(2))
70        CONTINUE
          IF(X2(1).GT.X2(2))THEN
             A(SUM)=X2(1)
             B(SUM)=X2(2)
          ELSE
             A(SUM)=X2(2)
             B(SUM)=X2(1)
          ENDIF
80        CONTINUE
90        CONTINUE
          CALL SORT
          IF(COUNT.LT.IREP)GOTO 97
          RETURN
          END
```

```
*****************************************************************
     *                                                      *
     *      This subroutine generates random numbers        *
     *          from the triangular distribution.           *
     *                                                      *
*****************************************************************
```

101

```
          SUBROUTINE TRIANG
          DIMENSION X(2),G(2),H(2),C(2),NSAMP(2)
          COMMON/ALG/A(50),B(50),SUM
          COMMON/ALL/NFILE,IREP,ITEM
          COMMON/FILE/NAME,DISP
          CHARACTER*13 NAME,DISP
          INTEGER COUNT,I1,SUM
          COUNT=0
          IF(NFILE.EQ.6)GOTO 9
          OPEN(8,FILE=NAME,STATUS='OLD',DISP=DISP)
    9     PRINT 50
    50    FORMAT(1X,'Enter minimum, maximum, mode and number of ',
         /  ' of samples from the',/,T2,
         /  'first triangular distribution separated by commas.')
          READ(5,*)G(1),H(1),C(1),NSAMP(1)
    13    PRINT 51
    51    FORMAT(1X,'Enter minimum, maximum, mode and number of ',
         /  ' of samples from the',/,T2,
         /  'second triangular distribution separated by commas.')
          READ(5,*)G(2),H(2),C(2),NSAMP(2)
          IF(G(1).GT.H(1).OR.G(1).GT.C(1).OR.C(1).GT.H(1))THEN
            WRITE(6,7)
    7       FORMAT(T2,'INCORRECT PARAMETERS ON  DISTRIBUTION.',
         /  ' TRY AGAIN.')
            GOTO 9
          ELSE IF(G(2).GT.H(2).OR.G(2).GT.C(2).OR.C(2).GT.H(2))THEN
            WRITE(6,7)
            GOTO 13
          ELSE
            PRINT 52
    52      FORMAT(T2,'Enter a five-digit number for random number',
         /  ' generation.')
            READ(5,*)I1
          END IF
          WRITE(NFILE,98)G(1),H(1),C(1),G(2),H(2),C(2),NSAMP(1),
         /  NSAMP(2),I1
    98    FORMAT(T2,'Input distributions are',
         /  ' T~(',F5.2,',',F5.2,',',F5.2,') and',
         /  ' T~(',F5.2,',',F5.2,',',F5.2,').',
         /  ,T2,'Number of samples from each distribution is ',
         /  I2,' and ',I2,' respectively.',
         /  /,T2,'The random number seed for this run is ',I6,'.')
    97    COUNT=COUNT+1
          SUM=0
          DO 30 I=1,2
          T=H(I)-G(I)
          COMP=(C(I)-G(I))/T
          COMP1=1.-COMP
          DO 20 L=1,NSAMP(I)
          SUM=SUM+1
          DO 10 J=1,2
          U=RAN(I1)
          IF(U.LE.COMP)THEN
            X1=COMP*U
            X2=SQRT(X1)
            X(J)=G(I)+T*X2
          ELSE
            X1=COMP1*(1.-U)
            X2=SQRT(X1)
            X3=1.-X2
```
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```
              X(J)=G(I)+I*X3
          ENDIF
    10    CONTINUE
          IF(X(1).GT.X(2))THEN
            A(SUM)=X(1)
            B(SUM)=X(2)
          ELSE
            A(SUM)=X(2)
            B(SUM)=X(1)
          ENDIF
    20    CONTINUE
    30    CONTINUE
          CALL SORT
          IF(COUNT.LT.IREP)GOTO 97
          RETURN
          END



***************************************************************************
        *                                                                 *
        *           This subroutine performs a bubble sort.               *
        *                                                                 *
***************************************************************************



          SUBROUTINE SORT
          DIMENSION AA(2),BB(2)
          COMMON/ALG/A(50),B(50),SUM
          COMMON/ALL/NFILE,IREP,ITEM
          COMMON/OUT/ARESULT(50),BRESULT(50),PERCENT(50)
          COMMON/FILE/NAME,DISP
          CHARACTER*13 NAME,DISP
          INTEGER ARUN,BRUN,SUM
    12    INTERA=1
          DO  I=1,(SUM-1)
           IF(A(I).GE.A(I+1))GOTO 25
             AA(2)=A(I)
             AA(1)=A(I+1)
             A(I)=AA(1)
             A(I+1)=AA(2)
             INTERA=0
    25     CONTINUE
          END DO
          IF(INTERA.EQ.0)GOTO 12
     9    INTERB=1
          DO J=1,(SUM-1)
           IF(B(J).GE.B(J+1))GOTO 27
             BB(2)=B(J)
             BB(1)=B(J+1)
             B(J)=BB(1)
             B(J+1)=BB(2)
             INTERB=0
    27     CONTINUE
          END DO
          IF(INTERB.EQ.0)GOTO 9
          ARUN=0
          BRUN=0
          DO I=1,SUM
           IF(A(I).GT.B(I))THEN
             ARUN=ARUN+1
```

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```
        ELSE
          GOTO 7
        END IF
        END DO
7       CONTINUE
        DO I=SUM,1,-1
         IF(B(I).LT.A(SUM))THEN
          BRUN=BRUN+1
         ELSE
          GOTO 18
         END IF
        END DO
18      CONTINUE

              ***************************************
              *     End of replication bookkeeping! *
              * This part of the subroutine stores  *
              * the statistic "# of runs of length  *
              *      X" in the array RESULT(X)       *
              ***************************************

        DO I=1,SUM
         IF(ARUN.EQ.I)ARESULT(I)=ARESULT(I)+1
         IF(BRUN.EQ.I)BRESULT(I)=BRESULT(I)+1
        END DO
        RETURN
        END

***********************************************************************
          *                                                    *
          *        This subroutine  generates the output of     *
          *                 the simulation.                     *
          *                                                    *
***********************************************************************


        SUBROUTINE REPORT
        DIMENSION AA(2),BB(2),CUM(50),INT(50)
        COMMON/ALG/A(50),B(50),SUM
        COMMON/ALL/NFILE,IREP,ITEM
        COMMON/OUT/ARESULT(50),BRESULT(50),PERCENT(50)
        COMMON/FILE/NAME,DISP
        CHARACTER*13 NAME,DISP,ANSWER
        INTEGER SUM
        IF(NFILE.EQ.6)GOTO 11
        OPEN(8,FILE=NAME,STATUS='OLD',DISP=DISP)
11      IF(ITEM.EQ.1.OR.ITEM.EQ.3)THEN
         ASSIGN 37 TO IOUT
        ELSE
         ASSIGN 39 TO IOUT
        END IF
        WRITE(NFILE,IOUT)
37      FORMAT(///,T2,'Test for the Length of the Initial Run of',
     /  ' A''s before the first B.',/)
39      FORMAT(///,T2,'Test for the Length of the Initial Run of ',
     /  'B''s before the first A.',/)
        WRITE(NFILE,13)IREP
13      FORMAT(T2,'Number of Replications: ',I6,/)
        IF(ITEM.EQ.1.OR.ITEM.EQ.3)THEN
         WRITE(NFILE,42)
42       FORMAT(T2,'Length ',4X,'Observed  ',4X,
```

```
   /    'Relative  ',4X,'Cumulative')
        WRITE(NFILE,44)
  44    FORMAT(T2,'of Run:',4X,'Frequency:',4X,
   /    'Frequency:',4X,'Frequency:')
        T=IREP
        CUM(0)=0.0
        DO I=1,SUM
         PERCENT(I)=ARESULT(I)/T
         CUM(I)=CUM(I-1)+PERCENT(I)
        END DO
        DO I=1,SUM
         INT(I)=ARESULT(I)
         WRITE(NFILE,17)I,INT(I),PERCENT(I),CUM(I)
  17     FORMAT(T2,I3,T12,I6,T28,F8.6,T42,F8.6)
        END DO
        WRITE(NFILE,62)
  62    FORMAT(T2,/,/,/,/)
        CALL GRAPH
        IF(ITEM.EQ.1)GOTO 57
        ITEM=2
        GOTO 11
        ELSE
         WRITE(NFILE,42)
         WRITE(NFILE,44)
         T=IREP
         CUM(0)=0.0
         DO I=1,SUM
          PERCENT(I)=BRESULT(I)/T
          CUM(I)=CUM(I-1)+PERCENT(I)
         END DO
         DO I=1,SUM
          INT(I)=BRESULT(I)
          WRITE(NFILE,17)I,INT(I),PERCENT(I),CUM(I)
         END DO
         WRITE(NFILE,62)
         CALL GRAPH
        END IF
  57    CONTINUE
        RETURN
        END


****************************************************************************
           *                                                     *
           *     This subroutine generates the relative and      *
           *        cumulative frequency distribution            *
           *                 function graphs.                    *
           *                                                     *
****************************************************************************


        SUBROUTINE GRAPH
        DIMENSION TEST1(50),CUM(50)
        COMMON/ALG/A(50),B(50),SUM
        COMMON/ALL/NFILE,IREP,ITEM
        COMMON/OUT/ARESULT(50),BRESULT(50),PERCENT(50)
        COMMON/FILE/NAME,DISP
        CHARACTER*13 NAME,DISP,ANSWER
        INTEGER SUM
        IF(NFILE.EQ.6)GOTO 13
        OPEN(8,FILE=NAME,STATUS='OLD',DISP=DISP)
```

```
13      IF(ITEM.EQ.2)THEN
          ASSIGN 99 TO L
          ASSIGN 88 TO K
          ASSIGN 84 TO M
        ELSE
          ASSIGN 98 TO L
          ASSIGN 86 TO K
          ASSIGN 82 TO M
        END IF
        WRITE(6,L)
98      FORMAT(T2,'For this test of the length of the run of',
     /  ' A''s before the first B,',/,T2,
     /  'do you wish a bar graph of the relative and',
     /  ' cumulative frequency distributions?',/,T2,
     /  'Enter "YES" or "NO".')
99      FORMAT(T2,'For this test of the length of the run of',
     /  ' B''s before the first A,',/,T2,
     /  'do you wish a bar graph of the relative and',
     /  ' cumulative frequency distributions?',/,T2,
     /  'Enter "YES" or "NO".')
        READ(5,91)ANSWER
91      FORMAT(A)
        IF(ANSWER.EQ.'NO')GOTO 89
        WRITE(NFILE,27)
27      FORMAT('1')
        WRITE(NFILE,K)
86      FORMAT(T5,'Relative frequency graph',
     /  ' for the Length of the Initial Run of',
     /  ' A''s before the first B.',/)
88      FORMAT(T5,'Relative frequency graph',
     /  ' for the Length of the Initial Run of',
     /  ' B''s before the first A.',/)
        WRITE(NFILE,37)
37      FORMAT(T5,'0.00',T10,'.04',T14,'.08',T18,'.12',T22,'.16',
     /T26,'.20',T30,'.24',T34,'.28',T38,'.32',T42,'.36',T46,'.40',
     /T50,'.44',T54,'.48',T58,'.52',T62,'.56',T66,'.60',T70,'.64',
     /T74,'.68',T78,'.72',T82,'.76',T86,'.80',T90,'.84',T94,'.88',
     /T98,'.92',T102,'.96',T106,'1.00')
31      WRITE(NFILE,33)
33      FORMAT(T6,'+-+-+-+-+-+-+-+-+-+-+-',
     /  '+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-',
     /  '+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+')
        CUM(0)=0.0
        DO I=1,SUM
         TEST1(I)=(PERCENT(I)+.005)*1000.
         J=TEST1(I)/10.
         IF(J.EQ.0)THEN
           ASSIGN 43 TO IOUT
         ELSE
           ASSIGN 41 TO IOUT
         END IF
         CUM(I)=CUM(I-1)+PERCENT(I)
         WRITE(NFILE,IOUT)I,PERCENT(I)
41       FORMAT('0',T3,I2,T6,'-',<J>(1H+),2X,F8.3)
43       FORMAT('0',T3,I2,T6,'-',2X,F8.3)
        END DO
        WRITE(NFILE,27)
        WRITE(NFILE,M)
82      FORMAT(T5,'Cumulative frequency graph',
     /  ' for the Length of the Initial Run of',
     /  ' A''s before the first B.',/)
```

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```fortran
84      FORMAT(I5,'Cumulative frequency graph',
     /  ' for the Length of the Initial Run of',
     /  ' B''s before the first A.',/)
        WRITE(NFILE,37)
        WRITE(NFILE,33)
        DO I=1,SUM
         TEST1(I)=(CUM(I)+.005)*1000.
         J=TEST1(I)/10.
         IF(J.EQ.0)THEN
          ASSIGN 43 TO IOUT
         ELSE
          ASSIGN 41 TO IOUT
         END IF
         WRITE(NFILE,IOUT)I,CUM(I)
        END DO
        WRITE(NFILE,27)
89      CONTINUE
        RETURN
        END
```

****

APPENDIX IV

ACKNOWLEDGEMENTS

APPENDIX V

REFERENCES

1. Downs and Cox, "The Probability of Motor Case Rupture," presented at the 20th Conference on Design of Experiments in Army R,D&T, October 1974.

2. Knaub, "Design of a Multiple Sample Westenberg Type Test for Small Sample Sizes," presented at the 27th Conference on Design of Experiments in Army R,D&T, October 1981.

3. Law and Kelton, Simulation Modeling and Analysis, McGraw-Hill Inc., 1982.

ADDENDUM

After listening to the presentation of this paper, Dr. W. J. Conover of Texas Tech, commented that perhaps a rank test based on the rank sum of the A's or some other appropriate measure might be used as a more powerful test of overall compatibility among the samples with emphasis on shift of location. In view of the weakness of the modified Westenberg test for the median which is given here, and the generally high power of rank tests, this suggestion seems promising. I would still, however, suggest the LIR test for the purpose for which it is intended: It emphasizes the improvement which can be obtained by exclusion of a particular sample. However, beware of repeated deletions. The probability levels of the test change step-wise with each application.